PERGAMON

International Journal of Solids and Structures 37 (2000) 3807–3839

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Determination of mechanism's order for kinematically and statically indetermined systems

N. Vassart, R. Laporte, R. Motro*

Laboratoire de Mécanique et de Génie Civil, Université Montpellier II, Montpellier Cedex, France

Received 9 July 1999

Abstract

This paper deals with the mechanisms in kinematically and statically indeterminate reticulated systems. Knowledge of length variation amplitude for members in association with an assigned mechanism allows determination of mechanism order. This is a fundamental characteristic of these systems, mainly for stability considerations. We submit on one part, simple tests allowing distinction between "order one mechanisms" and mechanisms of higher order, and on the other part an algorithm giving access to order exact value for all mechanisms associated with a given reticulated system. With this algorithm, order one mechanisms and higher order mechanisms are identified. Simple examples are given in the text and illustrate these aspects. In conclusion, we submit a stop criterion for the algorithm which gives access to the finite mechanisms for most of constructive reticulated systems. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Mechanism order; Infinitesimal mechanism; Finite mechanism; Reticulated system

1. Introduction

1.1. Equilibrium and compatibility equations

Reticulated systems are pin-jointed systems comprising members with bilateral or unilateral rigidity (bars, struts and cables) assembled with perfect pins. Members have a straight mean fibber, loads are applied on nodes. In case of coincidence of mean fibbers at the centre of gravity of a node, members are only subjected to tension or compression stresses. For a system with b

* Corresponding author. Fax: +33-4-67-14-45-55.
E-mail address: motro@lmgc.univ-montp2.fr (R. Motro).

PII: S0020-7683(99)00178-X

Nomenclature

$[A]$	equilibrium matrix $N \times b$ for a reticulated system in its reference configuration
$[A]^t$	transpose matrix of equilibrium matrix $[A]$ or compatibility matrix for reference configuration
b	number of members for a reticulated system
$\{d\}$	vector (N components) of node displacements related to reference configuration
$\{d^K\}$	mechanism vector, defined by $[A]^t\{d^K\} = \{0\}$ (i.e. $\{d^K\} \in \text{Ker } A^t$)
$\{d^{K\rho}\}$	vector belonging to mechanisms basis ($\{d^{K\rho}\} \in \text{Ker } A^t$)
$\{d^{K(a)}\}$	the "a" order part of a mechanism $\{d^K\}$ ($\{d^{K(a)}\} \in \text{Ker } A^t$)
$\{d^I\}$	vector nodes displacements, orthogonal to the mechanisms (i.e. $\{d^I\} \in \text{Im } A$)
$\{d^{Ip}\}$	vector belonging to orthogonal displacements basis ($\{d^{Ip}\} \in \text{Im } A$)
$\{d^{I(a)}\}$	the "a" order part of an orthogonal displacement ($\{d^{I(a)}\} \in \text{Im } A$)
$[D^0]$	connection matrix $N \times N$ of selfstress coefficients
$[\Delta A_d]$	matrix $N \times b$ corresponding to the matrix difference between the equilibrium matrix $[A^{\text{def}}(d)]$, in its deformed state defined by $\{d\}$, and the equilibrium matrix $[A]$ in its reference state: $[\Delta A_d] = [A^{\text{def}}(d)] - [A]$
$\{e\}$	vector (b components) of member length variation coefficients evaluated in respect to the reference configuration
e_j	length variation coefficient for member j, defined by $e_j = \ell_j^0(\ell_j - \ell_j^0)$
Im A	vectorial column space of the equilibrium matrix $[A]$ (dimension $= r_A$)
k	number of restrictions for node displacements for a reticulated system
Ker A	vectorial nullspace of the equilibrium matrix $[A]$
Ker A^t	vectorial left nullspace of the equilibrium matrix $[A]$ (dimension $= m$)
ℓ_j^0	member j length in reference state
ℓ_j	member j length in a loaded state
m	number of independent mechanisms for a reticulated system
n	number of nodes for a reticulated system
N	number of degrees of freedom (dof) for a reticulated system ($N = 3n - k$ for space systems), ($N = 2n - k$ for plane systems)
$\{q\}$	vector (b components) of force density coefficients
q_j	force density coefficient of member j, defined by $q_j = T_j/\ell_j^0$
$\{q^0\}$	selfstress vector for a reticulated system, defined by $[A]\{q^0\} = \{0\}$
q_j^0	selfstress coefficient of member j, defined by $q_j^0 = T_j^0/\ell_j^0$
r_A	rank of the equilibrium matrix $[A]$ (or of $[A]^t$)
T_j	axial compressive or tensile stress for member j
T_j^0	axial compressive or tensile stress for member j in reference state

elements. n nodes and k displacement restrictions for node, imposed by a frame, the number of degrees of freedom for the whole system is $N = 3n - k$ ($N = 2n - k$ for two-dimensional systems).

Kinematically and statically indeterminate reticulated systems are characterised by mechanisms and selfstress states. Pellegrino and Calladine (1986) developed a method which gives a selfstress state basis $\{q^0\}$ and a mechanism basis $\{d^K\}$. This basis determination relies on the study of the equilibrium matrix $[A]$ of the system in its reference configuration. Reference configuration is associated with the non loaded, but selfstressed geometry. Linear algebra results are applied to equilibrium and compatibility equations written in a matrix form as follows:

Equilibrium equation: $[A]\{q\} = \{f\}$ (1)

Compatibility equation: $[A]^{t}\{d\} = \{e\}$ (2)

where $[A]$ is the equilibrium $N \times b$ matrix (reference configuration).

$\{q\}$ is the b vector of force density coefficients:

For a member j: $q_j = \frac{T_j}{l_j^0}$ (3)

where T_j is the axial compressive or tensile stress for member j, l_j^0 is the length of member j in its reference state, $\{f\}$ is the N-vector of external loads applied on nodes, $[A]^t$ is the transpose of equilibrium matrix $[A]$, $\{d\}$ is the N-vector of node displacements related to reference configuration (these free displacements are assumed to be very small), $\{e\}$ is the b-vector (b components) of member length variation coefficients evaluated with respect to reference configuration:

For a member j: $e_j = l_j^0 \Delta l_j = l_j^0 \left(l_j - l_j^0 \right)$ (4)

and l_j is the length of member j in a loaded state.

We adopted this form for equilibrium and compatibility equations, using force density coefficients and member length variation coefficients, since it allows to split in the first member of $[A]\{q\} = \{f\}$, lengths (included in $[A]$) from products "mass · (time)$^{-2}$" (included in coefficients q_j).

1.2. Selfstress vector

In reference configuration, without external loads (i.e. $\{f\} = \{0\}$), vector $\{q\}$ is called selfstress vector and is noted $\{q^0\}$. It *exactly* verifies:

$$[A]\{q^0\} = \{0\} \quad (5)$$

A selfstress state basis $\{q^0\}$ is a basis of matrix $[A]$ nullspace ($\{q^0\} \in \mathrm{Ker}\, A$), which is a vectorial subspace of space R^b (members space).

We assume that materials are in their linear elastic range, which does not restrict the generality of results. In such a case, b elastic relations may be written:

$$\forall j = 1, \ldots, b: \quad q_j = q_j^0 + H_j e_j \text{ or in matrix form: } \quad \{q\} = \{q^0\} + [H]\{e\} \quad (6)$$

where H_j is the elastic coefficient of member j, $[H]$ is the $b \times b$ elasticity matrix, and $[H]$ is a positive definite diagonal matrix.

1.3. Mechanism vector

Particular values of $\{d\}$ for which $\{e\} = \{0\}$ at first order (length variation coefficients are equal to zero) can be found. These displacements, noted $\{d^K\}$, are called mechanisms and they are defined *at first order* by the relationship (derived from compatibility equations (2)):

$$[A]^t\{d^K\} = \{0\} \quad (7)$$

A mechanism basis $\{d^{\mathrm{K}}\}$ can be calculated by determination of the vectorial left nullspace of matrix $[A]^{\mathrm{t}}$ ($\{d^{\mathrm{K}}\} \in \mathrm{Ker}\, A^{\mathrm{t}}$). The m vectors of this basis will be noted $\{d^{\mathrm{K}\rho}\}$ ($\rho = 1, \ldots, m$).

In displacement space R^N, the vectorial subspace Im A (column space of $[A]$) is the orthogonal supplementary space of Ker A^{t}. Displacements which belong to this vectorial subspace Im A are noted $\{d^{\mathrm{I}}\}$ ($\{d^{\mathrm{I}}\} \in \mathrm{Im}\, A$) and are orthogonal to the mechanisms. These displacements create length variations in the members. Length variations are of same order as these displacements.

A general displacement $\{d\}$ can be so uniquely splitted in a displacement $\{d^{\mathrm{K}}\}$ belonging to Ker A^{t} and a displacement $\{d^{\mathrm{I}}\}$ belonging to Im A:

$$\{d\} = \left\{d^{\mathrm{K}}\right\} + \left\{d^{\mathrm{I}}\right\} \quad \text{since } R^N = \mathrm{Im}\, A \oplus \mathrm{Ker}\, A^{\mathrm{t}} \quad (\oplus = \text{direct summation}) \tag{8}$$

where $\{d\} \in R^N$, $\{d^{\mathrm{K}}\} \in \mathrm{Ker}\, A^{\mathrm{t}}$, $\{d^{\mathrm{I}}\} \in \mathrm{Im}\, A$, and with $\{d^{\mathrm{K}}\}^{\mathrm{t}} \bullet \{d^{\mathrm{I}}\} = 0$, "$\bullet$" is used for canonical scalar product in R^N.

1.4. Mechanism order definition

Mechanism's definition and its study are related to size order notions. Equivalent norm for R^N or R^b are used to quantify size orders. Euclidean norm for a vector $\{X\}$ of R^N will be noted $\|X\|$ and given by:

$$\|X\| = \left(\sum_{i=1}^{N} X_i^2 \right)^{1/2} \tag{9}$$

System member lengths ℓ_j are considered as finite, and assumed to be of zero order:

$$\forall j, \quad \ell_j = O_0 \quad \text{with } O_0 = \text{zero order} \tag{10}$$

or

$$\|\ell\| = O_0 \quad \text{with } \{\ell\} = \left\{\ell_1, \ldots, \ell_j, \ldots, \ell_b\right\}^{\mathrm{t}} \tag{11}$$

Applied loads on nodes are assumed to be so, that node displacements are small with respect to system size. Consequently, vector $\{d\}$ norm is assumed to be of order one:

$$\|d\| = O_1 \quad \text{with } O_1 = \xi O_0 \tag{12}$$

with ξ being a strictly positive real number very small with respect to one ($\xi \ll 1$).

Similarly, order "r" is defined by:

$$O_r = \xi O_{r-1} = \xi^r O_0 \tag{13}$$

Written form "$\overset{(O_r)}{\approx}$" will be used for equalities limited to order r. And "a" order value of X will be quoted $X^{(a)}$. Thus, it can be written as:

$$X = \sum_{a \geq a_0} X^{(a)} \quad \text{with } \|X^{(a)}\| = O_a \quad \text{or} \quad X^{(a)} = 0 \quad \text{(exact equality)} \tag{14}$$

These two possibilities are associated with the symbol "$\overset{0}{=}$" i.e.:

$$\|X^{(a)}\| \overset{0}{=} O_a \;\Leftrightarrow\; \|X^{(a)}\| = O_a \quad \text{or} \quad X^{(a)} = 0 \quad \text{(exact equality)} \tag{15}$$

The equality symbol "$=$", when used without superscript implies that term $X^{(a)}$ is not equal to zero

$$\|X^{(a)}\| = O_a = \xi^a O_0 \Longrightarrow X^{(a)} \neq 0 \tag{16}$$

This paper deals only with internal mechanisms of reticulated systems. Solid type mechanisms corresponding to overall displacements are previously identified (Pellegrino and Calladine, 1986) and considered separately. We make the hypothesis that at least one node is fixed and that the system is connected.

It is necessary to distinguish "first order infinitesimal mechanisms" from "higher order infinitesimal mechanisms". According to Koiter's definitions (Koiter, 1984): "an infinitesimal mechanism of the first order is characterised by its property that any infinitesimal displacement of the mechanism is accompanied by second-order elongations of at least some of the bars. An infinitesimal mechanism is called of second (or higher) order, if there exists an infinitesimal motion such that no bar undergoes an elongation of lower than the third (or higher) order".

To define more precisely the order of mechanisms, we use the formulation submitted by Tarnai (1984), formulation that we extended to multiparametered case, and which is similar to the one submitted by Salerno (1992). An internal mechanism is called mechanism of order "r" ($r \geq 1$) if there exists infinitesimal node displacements (of the first order) such as member length variations are equal to zero until order r, but there does not exist infinitesimal node displacements such as member length variations are equal to zero at order $r+1$:

Mechanism of order r $\Leftrightarrow$

$$\begin{cases} \exists \|d\| = O_1 \quad \left(\text{i.e. } \{d\} \neq \{0\}\right): \left\{e^{(1)}\right\} = \{0\}, \left\{e^{(2)}\right\} = \{0\}, \ldots, \{e^{(r)}\} = \{0\} \\ \forall \|d\| = O_1: \left\{e^{(r+1)}\right\} \neq \{0\} \end{cases} \tag{17}$$

A mechanism is called finite mechanism if there exists a displacement which does not generate length variations of any order.

Many authors have worked recently on mechanism's order determination for kinematically indeterminate systems.

Calladine and Pellegrino (1991a, 1992) submit a test, which is based on energetical computations and which allows to establish a distinction between mechanisms of the first order and of higher order. Kuznetsov (1988, 1991a, 1991b, 1991c) develops a method based on the decomposition of the system in sub-systems. Tarnai (1989) uses a geometrical method, with which he tests all the possible displacements in order to find (by a max (min) research) those which are associated with the least length variations. But this method can only be used for simple or periodic reticulated systems.

Lastly Salerno (1992) gives a numerical method based on energetical properties of systems. The corresponding algorithm is based on the calculation of deformation energy of system supposed to be in zero selfstress state, and length variations for members appear in a quadratic form. In this method after a parametring operation, energy is developed as a series, whose increasing order terms are minimised. These calculations are done in the vicinity of mechanisms, but without explicit decomposition of displacements in two orthogonal vectorial subspaces of R^N ($R^N = \text{Im}\, A \otimes \text{Ker}\, A^t$). Corresponding results, given in numerical form, give only an inferior limit of mechanism's order, certainly because of calculation complexity.

We describe in this paper an analytic method for which only geometrical properties of kinematically indeterminate systems are taken into account. With this method and mainly with the underlying algorithm (Section 5), order of infinitesimal mechanisms can be evaluated without limitation for order's level. A stop criterion (Section 7) is also given so as to detect possible finite mechanisms of a kinematically indeterminate system.

Before giving a description of our algorithm, higher order mechanisms (Section 2) and first-order mechanisms (Section 3) are characterised geometrically and energetically. Some examples illustrate simultaneously these characteristics and corresponding associated methods (Sections 4, 6 and 7.3).

2. Higher order mechanism characterisation

2.1. Higher order mechanism geometrical characterisation

According to the definition, a reticulated system admits a higher order mechanism, if there exists a displacement $\{d\}$ of order one (i.e. not equal to zero), such as length variation coefficients are equal to zero until order two, i.e.:

$$\exists \|d\| = O_1 \quad \text{such as } \{e\} \overset{(O_2)}{\approx} \{0\} \tag{18}$$

To express this geometrical characteristic, it is interesting to write the next relationship between length difference for a member j, from its reference state (ℓ_j^0) to its deformed state (ℓ_j), and displacements of it extremities i and h:

$$(\ell_j)^2 - \left(\ell_j^0\right)^2 = \sum_{x,\,y,\,z} \left[x_i - x_h + d_{ix}^{\mathrm{K}} - d_{hx}^{\mathrm{K}} + d_{ix}^{\mathrm{I}} - d_{hx}^{\mathrm{I}}\right]^2 - \sum_{x,\,y,\,z} (x_i - x_h)^2 \tag{19}$$

with

$$(\ell_j)^2 - \left(\ell_j^0\right)^2 = \left(\ell_j^0 + \Delta\ell_j\right)^2 - \left(\ell_j^0\right)^2 = 2e_j + \frac{e_j^2}{\left(\ell_j^0\right)^2} \tag{20}$$

where d_{ix}^{K} is the component of the mechanism vector $\{d^{\mathrm{K}}\}$, associated with node i - degree of freedom along X direction (and so it is for d_{ix}^{I} with vector $\{d^{\mathrm{I}}\}$).

Then the following exact expression may be established:

$$e_j + \frac{e_j^2}{2\left(\ell_j^0\right)^2} = \sum_{x,\,y,\,z} \left[(x_i - x_h)\left(d_{ix}^{\mathrm{I}} - d_{hx}^{\mathrm{I}}\right) + \frac{1}{2}\left(d_{ix}^{\mathrm{K}} - d_{hx}^{\mathrm{K}}\right)^2 + \left(d_{ix}^{\mathrm{K}} - d_{hx}^{\mathrm{K}}\right)\left(d_{ix}^{\mathrm{I}} - d_{hx}^{\mathrm{I}}\right) + \frac{1}{2}\left(d_{ix}^{\mathrm{I}} - d_{hx}^{\mathrm{I}}\right)^2\right] \tag{21}$$

A matrix form of this equation is obtained by noting $[\Delta A_d]$ the matrix difference between the equilibrium matrix $[A^{\mathrm{def}}(d)]$, in its deformed state under $\{d\}$, and the equilibrium matrix $[A]$ in its reference state:

$$[\Delta A_d] = \left[A^{\mathrm{def}}(d)\right] - [A] \tag{22}$$

Terms of $[\Delta A_d]$ are simply obtained by replacing coordinates x_i in matrix $[A]$ by corresponding displacements d_{ix}.

Two matrices $[\Delta A_{d^{\mathrm{K}}}]$ and $[\Delta A_{d^{\mathrm{I}}}]$ associated with displacements $\{d^{\mathrm{K}}\}$ and $\{d^{\mathrm{I}}\}$ are introduced and relationship (21) becomes the following *exact* matrix expression:

$$\left\{ e_j + \frac{e_j^2}{2\left(\ell_j^0\right)^2} \right\} = [A]^t\{d^I\} + \frac{1}{2}\left[\Delta A_{d^K}\right]^t\{d^K\} + \left[\Delta A_{d^K}\right]^t\{d^I\} + \frac{1}{2}\left[\Delta A_{d^I}\right]^t\{d^I\} \tag{23}$$

Geometrical characterisation requires a study in the vicinity of a mechanism: we assume that external loads create mechanism displacements $\{d^K\}$ of order one and orthogonal displacements $\{d^I\}$ of order greater or equal to two:

$$\{d\} = \{d^K\} + \{d^I\} \quad \text{with } \|d\| = O_1,\ \|d^K\| = O_1 \text{ and } \|d^I\| \le O_2. \tag{24}$$

According with our hypothesis, we may also write:

$$\|[A]^t\{d^I\}\| \le O_2,\ \|\left[\Delta A_{d^K}\right]^t\{d^I\}\| \le O_3 \text{ and } \|\left[\Delta A_{d^I}\right]^t\{d^I\}\| \le O_4 \tag{25}$$

But, since it is assumed that the system is connected and that at least one node is fixed, the product $[\Delta A_{d^K}]^t\{d^K\}$ is of order O_2 when the mechanism vector $\{d^K\}$ is of order O_1:

$$\|d^K\| = O_1 \Longrightarrow \|\left[\Delta A_{d^K}\right]^t\{d^K\}\| = O_2 \tag{26}$$

Consequently, if in relationship (23) we keep only main terms of order less than or equal to two, it remains:

$$\{e\} \overset{(O_2)}{\approx} [A]^t\{d^I\} + \frac{1}{2}\left[\Delta A_{d^K}\right]^t\{d^K\} \tag{27}$$

As the term $[A]^t\{d^I\}$ is of same order as vector $\{d^I\}$, one can cancel the sum $[A]^t\{d^I\} + (1/2)[\Delta A_{d^K}]^t\{d^K\}$ only if displacements $\{d^I\}$ are of order O_2. Therefore, we make this hypothesis:

$$\|d^I\| = O_2 \tag{28}$$

In this case, the two terms of the preceding expression are of order two, and it can be written:

$$\{e\} \overset{(O_2)}{\approx} [A]^t\{d^I\} + \frac{1}{2}\left[\Delta A_{d^K}\right]^t\{d^K\} = \{e^{(2)}\} \tag{29}$$

The following geometrical characterisation of a higher order mechanism may so be stated:

"A reticulated system admits a higher order mechanism, if, and only if, there exists in the vicinity of a mechanism $\{d^K\}$ (not equal to zero), a displacement $\{d^I\}$ ($\in$Im A) such as length variation coefficients $\{e\}$ are equal to zero until order two", which may be written as:

$$\exists\left(\{d^K\}, \{d^I\}\right) \in (\text{Ker } A^t - \{0\} \times \text{Im } A) \text{ such as } \{e^{(2)}\} = [A]^t\{d^I\} + \frac{1}{2}\left[\Delta A_{d^K}\right]^t\{d^K\} = \{0\} \tag{30}$$

2.2. Energetic characterisation of higher mechanisms

Strain energy W_j, for a member j of a reticulated system, between its reference state (not loaded) and the deformed state, when loaded, is defined by:

$$W_j = \int_{\ell_j^0}^{\ell_j} (T_j)\, \mathrm{d}\lambda \tag{31}$$

By introducing in the previous relationships, the force density coefficient q, the axial stress T_j may be written as:

$$T_j = q_j \ell_j^0 = \left(q_j^0 + H_j e_j\right)\ell_j^0 = q_j^0 \ell_j^0 + H_j\left(\ell_j^0\right)^2\left(\ell_j - \ell_j^0\right) \tag{32}$$

So, strain energy W_j is exactly:

$$W_j = q_j^0 e_j + \frac{1}{2} H_j e_j^2 \tag{33}$$

Corresponding energy for the whole system is written as:

$$W = \sum_{j=1}^{b} W_j = \{e\}^{\mathrm{t}}\left\{q^0\right\} + \frac{1}{2}\{e\}^{\mathrm{t}}[H]\{e\} \tag{34}$$

In the vicinity of a mechanism $\{d^{\mathrm{K}}\}$, length variation coefficients $\{e\}$ may be replaced by their second-order expressions (29). If terms of order less than or equal to two are only considered, it remains:

$$W \overset{(O_2)}{\approx} \left\{d^{\mathrm{I}}\right\}^{\mathrm{t}}[A]\left\{q^0\right\} + \frac{1}{2}\left\{d^{\mathrm{K}}\right\}^{\mathrm{t}}\left[\Delta A_{d^{\mathrm{K}}}\right]\left\{q^0\right\} \tag{35}$$

Indeed, it is assumed by hypothesis, that elastic coefficients H_j are of order O_{-1} (with $O_{-1} = O_0/\xi$, that selfstress coefficients $q_j^0 = H_j \Delta\ell_j^0$ are of order O_0 if $\mathrm{Ker}\, A \neq \{0\}$, and therefore, that length variations $\Delta\ell_j^0$ (between the non convened state and the reference state), creating this selfstress, are of order O_1.

But the product $[A]\{q^0\}$ is exactly equal to zero according to the definition of $\{q^0\}$, and strain energy evaluated at second order takes the form:

$$W \overset{(O_2)}{\approx} \frac{1}{2}\left\{d^{\mathrm{K}}\right\}^{\mathrm{t}}\left[\Delta A_{d^{\mathrm{K}}}\right]\left\{q^0\right\} \tag{36}$$

This form is the same as the one given by Calladine and Pellegrino (1991a, 1991b) for strain energy expression.

"Energetic" and "geometric" characterisation are equivalent criteria for higher order mechanisms. Indeed, multiplying Eq. (30) by any selfstress state $\{q^0\}$ belonging to Ker A, leads to the following expression:

$$\left\{e^{(2)}\right\}^{\mathrm{t}}\left\{q^0\right\} = \left([A]^{\mathrm{t}}\left\{d^{\mathrm{I}}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{K}}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}}\right\}\right)^{\mathrm{t}}\left\{q^0\right\} = 0 \tag{37}$$

hence,

$$\left\{d^{\mathrm{I}}\right\}^{\mathrm{t}}[A]\left\{q^0\right\} + \frac{1}{2}\left\{d^{\mathrm{K}}\right\}^{\mathrm{t}}\left[\Delta A_{d^{\mathrm{K}}}\right]\left\{q^0\right\} = 0 \tag{38}$$

so

$$\frac{1}{2}\left\{d^{\mathrm{K}}\right\}^{\mathrm{t}}\left[\Delta A_{d^{\mathrm{K}}}\right]\left\{q^0\right\} = 0 \quad \text{since } [A]\left\{q^0\right\} = \{0\} \tag{39}$$

Specific case corresponding to $\mathrm{Ker}\, A = \{0\}$ does not make an exception. For all systems without selfstress states, mechanisms orders are higher than one. In fact, one knows that all statically determinate and kinematically indeterminate systems are finite mechanisms. (We give otherwise in Section 7, a method of determination of finite mechanisms).

The converse proposal is also true. Indeed, if:

$$\exists\{d^{\mathrm{K}}\} \in (\mathrm{Ker}\, A^{\mathrm{t}} - \{0\}) \quad \text{such as } \forall\{q^0\} \in \mathrm{Ker}\, A,\ \frac{1}{2}\{d^{\mathrm{K}}\}^{\mathrm{t}}[\Delta A_{d^{\mathrm{K}}}]\{q^0\} = 0 \tag{40}$$

Then, since $R^b = \mathrm{Im}\, A^{\mathrm{t}} \oplus \mathrm{Ker}\, A$ and $\mathrm{Im}\, A^{\mathrm{t}} \perp \mathrm{Ker}\, A$, previous hypothesis implies that:

$$\frac{1}{2}[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} \in \mathrm{Im}\, A^{\mathrm{t}} \tag{41}$$

Thus, if $[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} \in \mathrm{Im}\, A^{\mathrm{t}}$, there exists $\{d\} \in R^N$ such as:

$$\frac{1}{2}[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} = -[A]^{\mathrm{t}}\{d\} \tag{42}$$

As $\{d\} = \{d'^{\mathrm{K}}\} + \{d^{\mathrm{I}}\}$ (with $\{d'^{\mathrm{K}}\}$ being an arbitrary mechanism) so:

$$\frac{1}{2}[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} = -[A]^{\mathrm{t}}\{d^{\mathrm{I}}\} \tag{43}$$

where $\{d^{\mathrm{I}}\} \in \mathrm{Im}\, A$ is necessarily of order two, if $\{d^{\mathrm{K}}\}$ is of order one.

Therefore,

$$[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} \in \mathrm{Im}\, A^{\mathrm{t}} \text{ is equivalent to:} \quad [A]^{\mathrm{t}}\{d^{\mathrm{I}}\} + \frac{1}{2}[\Delta A_{d^{\mathrm{K}}}]^{\mathrm{t}}\{d^{\mathrm{K}}\} = \{0\} \tag{44}$$

A higher order mechanism may be characterised as follows:

"A reticulated system admits a mechanism $\{d^{\mathrm{K}}\}$ (not equal to zero) and of order higher than one if and only if whatever can be its selfstress states, strain energy is, in the mechanism vicinity, always zero until order two":

$$\exists\{d^{\mathrm{K}}\} \in (\mathrm{Ker}\, A^{\mathrm{t}} - \{0\}), \quad \forall\{q^0\} \in \mathrm{Ker}\, A{:}\, W \overset{(O_2)}{\approx} \frac{1}{2}\{d^{\mathrm{K}}\}^{\mathrm{t}}[\Delta A_{d^{\mathrm{K}}}]\{q^0\} = 0 \tag{45}$$

This may also be written in the following quadratic form:

$$\exists\{d^{\mathrm{K}}\} \in (\mathrm{Ker}\, A^{\mathrm{t}} - \{0\}), \quad \forall\{q^0\} \in \mathrm{Ker}\, A{:}\, W \overset{(O_2)}{\approx} \frac{1}{2}\{d^{\mathrm{K}}\}^{\mathrm{t}}[D^0]\{d^{\mathrm{K}}\} = 0 \tag{46}$$

where $[D^0]$ is the $N \times N$ connection matrix of selfstress coefficients.

Indeed, there exists a symmetric matrix $[D^0]$ such as:

$$[\Delta A_{d^{\mathrm{K}}}]\{q^0\} = [D^0]\{d^{\mathrm{K}}\} \tag{47}$$

This matrix $[D^0]$ may be established as follows:

- $D^0_{ixix} = \sum_{j]i} q^0_j$ (summation of all selfstress coefficients for members linked to node i),
- $D^0_{ixhx} = -q^0_j$ if the nodes i and h ($h \neq i$) are connected by member j,
- $D^0_{ixhx} = 0$ else,
- $D^0_{ixhy} = 0\ \forall i, \forall h$ (i.e. when degrees of freedom ix and hy do not correspond to the same direction).

For free nodes, along the same direction, this leads to the matrix used by Sheck (1974):

$$[D^0] = [C_l]^t[Q^0][C_l] \tag{48}$$

where $[Q^0]$ is the diagonal matrix comprising the b selfstress coefficients and $[C_l]$ is the free nodes connectivity matrix of the system.

3. Characterisation of first-order mechanisms

3.1. Geometrical characterisation of order one mechanisms

A mechanism $\{d^K\}$, not equal to zero, is of order one, if it induces length variations of order two in system members. That is to say that, whatever the displacements $\{d^I\}$ orthogonal to the mechanisms are, the length variation coefficients vector remains different from zero at order two, in the vicinity of a mechanism $\{d^K\}$:

$$\forall\{d^I\} \in \text{Im } A{:}\{e^{(2)}\} = [A]^t\{d^I\} + \frac{1}{2}[\Delta A_{d^K}]^t\{d^K\} \neq \{0\} \tag{49}$$

Consequently, an indeterminate reticulated system admits only first-order mechanisms if and only if:

$$\forall(\{d^K\}, \{d^I\}) \in (\text{Ker } A^t - \{0\} \times \text{Im } A){:}\{e^{(2)}\} = [A]^t\{d^I\} + \frac{1}{2}[\Delta A_{d^K}]^t\{d^K\} \neq \{0\} \tag{50}$$

3.2. Energetic characterisation of first-order mechanisms

In this case it is sufficient to use the logical converse proposal of higher order mechanisms. An indeterminate reticulated system admits only first-order mechanisms if and only if:

$$\forall\{d^K\} \in (\text{Ker } A^t - \{0\} \times \text{Im } A), \quad \exists\{q^0\} \in \text{Ker } A{:}W \overset{(O_2)}{\approx} \frac{1}{2}\{d^K\}^t[D^0]\{d^K\} \neq 0 \tag{51}$$

So, again, if $\text{Ker } A = \{0\}$, it is found that possible mechanisms of a system can not be first-order mechanisms. In fact, these are finite mechanisms.

4. Application to simple reticulated systems

4.1. Reticulated system with two members assembled side by side

We consider a plane reticulated system, comprising two members of same length, assembled side by side (Fig. 1(a)). Analysis of equilibrium $[A] = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ allows to obtain a mechanism basis $\{d^K\} = \mu\{0, 1\}^t$, orthogonal displacements $\{d^I\} = u\{1, 0\}^t$ and selfstress states $\{q^0\} = \alpha\{1, -1\}^t$ (μ, u and α are arbitrary real numbers). We want to determine the order of mechanism $\{d^K\}$.

4.1.1. First method: geometrical characterisation

We calculate the length variation coefficients at order two in the vicinity of mechanism $\{d^K\}$ (Fig. 1(b)). According to relationship (29), we may write:

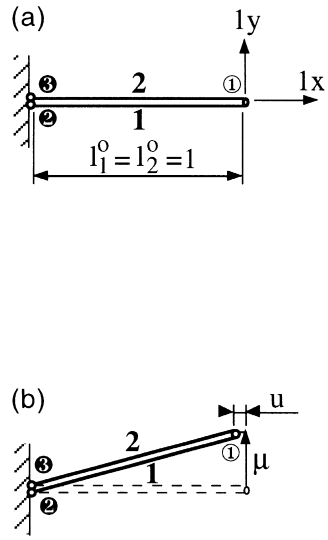

Fig. 1. (a) Plane reticulated system with two "side to side" members, in its reference state. (b) Plane reticulated system with two "side to side" members in its deformed state, in mechanism's vicinity.

$$\begin{cases} e_1 \overset{(O_2)}{\approx} u + \dfrac{\mu^2}{2} \\ e_2 \overset{(O_2)}{\approx} u + \dfrac{\mu^2}{2} \end{cases} \quad \text{since } \left[\Delta A_{d^{\mathrm{K}}}\right] = \begin{bmatrix} 0 & 0 \\ \mu & \mu \end{bmatrix} \tag{52}$$

We observe that there exists a displacement $\{d^{\mathrm{I}}\}$ (corresponding to $u = -\mu^2/2$) which cancels at order two the length variation coefficients. Then $\{d^{\mathrm{K}}\}$ is a mechanism of order higher than one.

Finally, this is a finite mechanism, since all the length variation coefficients $\{e\}$ may be cancelled for every order with $u = \sqrt{1-\mu^2} - 1$.

4.1.2. Second method: energetic characterisation

The connection matrix of selfstress coefficients is evaluated in order to obtain strain energy value at second order in the mechanism vicinity:

$$[D^0] = \begin{bmatrix} q_1^0 + q_2^0 & 0 \\ 0 & q_1^0 + q_2^0 \end{bmatrix} \Longrightarrow W \overset{(O_2)}{\approx} \frac{1}{2}\{d^{\mathrm{K}}\}^{\mathrm{t}}[D^0]\{d^{\mathrm{K}}\} = \left(q_1^0 + q_2^0\right)\frac{\mu^2}{2} = (\alpha - \alpha)\frac{\mu^2}{2} = 0 \tag{53}$$

Strain energy is always equal to zero until order two, independently of applied selfstress states. Mechanism $\{d^{\mathrm{K}}\}$ is a higher order mechanism.

4.2. Reticulated system with two members assembled end to end

Let us consider now, a plane reticulated system comprising two members of identical length, assembled end to end (Fig. 2(a)). Analysis of equilibrium matrix $[A] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ leads to the determination of a mechanism basis $\{d^{\mathrm{K}}\} = \mu\{0, 1\}^{\mathrm{t}}$, orthogonal displacements $\{d^{\mathrm{I}}\} = \upsilon\{0, 1\}^{\mathrm{t}}$, and selfstress states $\{q^0\} = \alpha\{1, 1\}^{\mathrm{t}}$.

Which is the order of a non zero mechanism $\{d^{\mathrm{K}}\}$ (i.e. $\mu \neq 0$)? Is it a first-order mechanism?

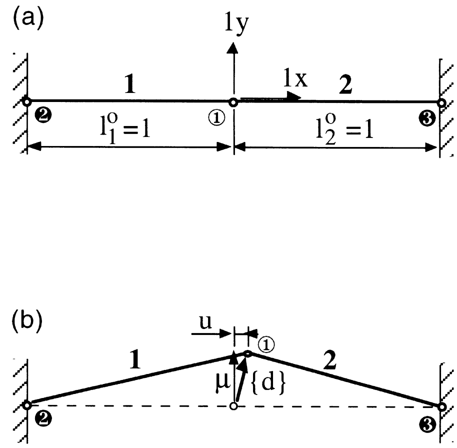

Fig. 2. (a) Plane reticulated system with two "end to end" members in its reference state. (b) Plane reticulated system with two "end to end" members in its deformed state, in mechanism's vicinity.

4.2.1. First method: geometrical characterisation

Vector $\{e\}$ is calculated until second order, in the vicinity of mechanism $\{d^{\mathrm{K}}\}$ (Fig. 2(b)). According to relationship (29), we get:

$$\begin{cases} e_1 \overset{(O_2)}{\approx} u + \dfrac{\mu^2}{2} \\ e_2 \overset{(O_2)}{\approx} -u + \dfrac{\mu^2}{2} \end{cases} \quad \text{since } \left[\Delta A_{d^{\mathrm{K}}}\right] = \begin{bmatrix} 0 & 0 \\ \mu & \mu \end{bmatrix} \tag{54}$$

The length variation coefficients e_1 and e_2 are not equal to zero, independently of the value u. None displacement $\{d^{\mathrm{I}}\}$ cancels $\{e\}$ at second order. Mechanism $\{d^{\mathrm{K}}\}$ is associated with length variations of second order, therefore, it is a first-order mechanism.

4.2.2. Second method: energetic characterisation

$$[D^0] = \begin{bmatrix} q_1^0 + q_2^0 & 0 \\ 0 & q_1^0 + q_2^0 \end{bmatrix} \Longrightarrow W \overset{(O_2)}{\approx} \frac{1}{2}\{d^{\mathrm{K}}\}^{\mathrm{t}}[D^0]\{d^{\mathrm{K}}\} = \left(q_1^0 + q_2^0\right)\frac{\mu^2}{2} = \alpha\mu^2 \tag{55}$$

Strain energy W can be equal to zero, for at least one selfstress state (corresponding to any $\alpha \neq 0$). So mechanism $\{d^{\mathrm{K}}\}$ is a first-order mechanism.

5. Order of infinitesimal mechanisms

5.1. Notations and hypothesis

The following algorithm leads to the determination of all higher mechanisms for a reticulated system and of the order value for a given infinitesimal mechanism.

It is based on geometrical characteristics in the mechanism vicinity and on splitting up general displacement $\{d\}$ in the two vectorial subspaces, complementary and orthogonal subspaces Ker A^{t} and Im A. So a displacement $\{d\}$ of order one may be written, in the mechanism vicinity $\{d^{\mathrm{K}}\}$ as:

$$\{d\} = \{d^{\mathrm{K}}\} + \{d^{\mathrm{I}}\} \text{ with } \{d^{\mathrm{K}}\} \in (\text{Ker } A^{\mathrm{t}} - \{0\}), \{d^{\mathrm{I}}\} \in \text{Im } A \text{ and } \|d^{\mathrm{K}}\| = O_1, \|d^{\mathrm{I}}\| = O_2 \tag{56}$$

It is very important to notice that the geometrical characterisation of higher order mechanism defines this mechanism only at first order. It gives only the order one main part of mechanism $\{d^{\mathrm{K}}\}$, which is noted $\{d^{\mathrm{K}(1)}\}$. Geometrical characterisation may be written too:

$$\exists\left(\{d^{\mathrm{K}(1)}\}, \{d^{\mathrm{I}(2)}\}\right) \in (\text{Ker } A^{\mathrm{t}} - \{0\} \times \text{Im } A): \{e\} \overset{(O_2)}{\approx} [A]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} + \frac{1}{2}[\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{d^{\mathrm{K}(1)}\} = \{0\} \tag{57}$$

where $\{d^{\mathrm{I}(2)}\}$ is the second-order main part of displacements $\{d^{\mathrm{I}}\}$.

This equation is used to identify first-order mechanisms. For subsequent steps, higher order terms of $\{d^{\mathrm{K}}\}$, and generally of $\{d^{\mathrm{I}}\}$ too, are taken into account so as to know if length variations cancel for a given order.

The method is based on a step by step algorithm: if there exists mechanisms of order higher than "r", the problem is to know if there are of order "$r+1$" and so on.

5.2. Mechanisms of first order or of higher order

First step of algorithm is based on the geometrical characterisation of higher order mechanism (written under the form (57)). It determines the possible higher order mechanism:

$$\begin{aligned} &\exists\left(\{d^{\mathrm{K}(1)}\}, \{d^{\mathrm{I}(2)}\}\right) \in (\text{Ker } A^{\mathrm{t}} - \{0\} \times \text{Im } A) \\ &\quad \text{such as } \{e^{(2)}\} = [A]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} + \frac{1}{2}[\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{d^{\mathrm{K}(1)}\} = \{0\} \end{aligned} \tag{58}$$

where $\{d^{\mathrm{K}(1)}\}$ is the first-order main part of $\{d^{\mathrm{K}}\}$, defined according to a mechanisms basis and $\{d^{\mathrm{I}(2)}\}$ is the second-order main part of $\{d^{\mathrm{I}}\}$, defined according to an orthogonal displacements basis.

If system of equations $\{e^{(2)}\} = \{0\}$ has no solution in $\{d^{\mathrm{K}(1)}\}$ and $\{d^{\mathrm{I}(2)}\}$, except zero solution, then the reticulated system admits only first-order mechanisms. Conversely, there exists at least one higher order mechanism, since there exists at least a displacement $\{d^{\mathrm{I}(2)}\}$ which cancels at order two length variation coefficients $\{e^{(2)}\}$ generated by the displacement $\{d^{\mathrm{K}(1)}\}$. At the end of this step, only the first-order main part $\{d^{\mathrm{K}(1)}\}$ of mechanism $\{d^{\mathrm{K}}\}$ is known.

With respect to $\{d^{\mathrm{I}(2)}\}$ components, the system is linear, but it is quadratic with respect to those of $\{d^{\mathrm{K}(1)}\}$.

To write the equations with independent variables, the decomposition of displacement vectors on a basis of their respective vectorial subspaces is used. Mechanism vectors are then defined as a linear combination of vectors $\{d^{\mathrm{K}1}\},\ldots,\{d^{\mathrm{K}\rho}\},\ldots,\{d^{\mathrm{K}m}\}$ which constitute a basis of the vectorial subspace Ker A^{t}:

$$\{d^{\mathrm{K}}\} = \sum_{\rho=1}^{m} \mu_\rho \{d^{\mathrm{K}\rho}\} \tag{59}$$

with $\|d^{\mathrm{K}\rho}\| = O_0$ and $|\mu_\rho| \le O_1$.

The vector comprising all m independent variables μ_ρ is noted as $\{\mu\}$:

$$\{\mu\} = \{\mu_1, \ldots, \mu_\rho, \ldots, \mu_m\}^{\mathrm{t}} \tag{60}$$

Similarly, we may express any displacement vector $\{d^{\mathrm{I}}\}$ in a basis $\{d^{\mathrm{I}1}\}, \ldots, \{d^{\mathrm{I}p}\}, \ldots, \{d^{\mathrm{I}r_A}\}$ of the vectorial subspace Im A as:

$$\{d^{\mathrm{I}}\} = \sum_{p=1}^{r_A} u_p \{d^{\mathrm{I}p}\} \tag{61}$$

with

$$\|d^{\mathrm{I}p}\| = O_0,\ |u_p| \le O_2 \text{ and } \{u\} = \{u_1, \ldots, u_p, \ldots, u_{r_A}\}^{\mathrm{t}} \tag{62}$$

It may be also written as:

$$\{d^{\mathrm{K}(1)}\} = \sum_{\rho=1}^{m} \mu_\rho^{(1)} \{d^{\mathrm{K}\rho}\} \quad \text{and} \quad \{d^{\mathrm{I}(2)}\} = \sum_{p=1}^{r_A} u_p^{(2)} \{d^{\mathrm{I}p}\} \tag{63}$$

It is then necessary to solve a system with b equations and $m + r_A = N$ unknowns ($\{\mu^{(1)}\}$ and $\{\mu^{(2)}\}$).

5.3. Order two mechanisms

If the considered system admits higher mechanisms, the question is to know if they are only of order two, or if they are the first terms of higher mechanisms. That is why, order three length variation coefficients $\{e^{(3)}\}$ are evaluated by taking into account higher order displacement terms. So, we write:

$$\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\} + \{D^{\mathrm{K}}\} \quad \text{with } \|D^{\mathrm{K}}\| \stackrel{0}{=} O_2 \tag{64}$$

and

$$\{d^{\mathrm{I}}\} = \{d^{\mathrm{I}(2)}\} + \{D^{\mathrm{I}}\} \quad \text{with } \|D^{\mathrm{I}}\| \stackrel{0}{=} O_3 \tag{65}$$

where $\{d^{\mathrm{K}(1)}\}$ and $\{d^{\mathrm{K}(2)}\}$ are the results of the previous step, $\{D^{\mathrm{K}}\}$ and $\{D^{\mathrm{I}}\}$ are the new unknowns (with $D^{\mathrm{K}} \in \mathrm{Ker}\, A^{\mathrm{t}}$ and $D^{\mathrm{I}} \in \mathrm{Im}\, A$) with:

$$\{D^{\mathrm{K}}\} = \sum_{\rho=1}^{m} \bar{\mu}_\rho \{d^{\mathrm{K}\rho}\} \quad \text{and} \quad \{D^{\mathrm{I}}\} = \sum_{p=1}^{r_A} U_p \{d^{\mathrm{I}p}\} \tag{66}$$

In the specific case related to only given mechanism $\{d^{\mathrm{K}}\}$ study (for example if $m = 1$), only components of displacements $\{d^{\mathrm{I}}\}$ may be used to cancel member length variations. Then, the algorithm is performed without modifying $\{d^{\mathrm{K}}\}$ (i.e. $\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\}$ since $\{D^{\mathrm{K}}\} = \{0\}$.)

If, in the exact relationship (23), between length variation coefficients and displacement vectors only preponderant terms of order three are considered, general writing of order three length variation coefficients is then:

$$\{e^{(3)}\} = [A]^{\mathrm{t}}\{D^{\mathrm{I}}\} + [\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{D^{\mathrm{K}}\} + [\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} \tag{67}$$

Remark. In this relationship no term related to term $e_j^2/2(\ell_j^0)^2$ included in exact relationship (22) appears, since it leads to order four or more than four terms, when length variations are equal to zero at order two. And so it is for all orders.

System of equations $\{e^{(3)}\} = \{0\}$ is a linear system with b equations and N unknowns ($\{\bar{\mu}$ and $\{U\}$). If it has no solution, the considered mechanism is of order two. In case of existence of at least a solution ($\{\bar{\mu}\}, \{U\}$), solution vectors of ($\{D^{\mathrm{K}}\}, \{D^{\mathrm{I}}\}$), noted ($\{d^{\mathrm{K}(2)}\}$, $\{d^{\mathrm{I}(3)}\}$) may be determined. Corresponding mechanisms $\{d^{\mathrm{K}}\} = (\{d^{\mathrm{K}(1)}\} + \{d^{\mathrm{K}(2)}\})$ are of order higher than two, and next step of algorithm has to be performed.

5.4. Order "r" mechanism

The algorithm may be generalised as follows.

Objective of step r ($r > 1$) in the algorithm is to determine if the mechanism found at the previous step, is only of order "r" or if it is the first term of higher order mechanism. After step $r-1$, displacements ($\{d^{\mathrm{K}^*}\}$, $\{d^{\mathrm{I}^*}\}$) are known, they cancel length variations until order r:

$$\exists\left(\left\{d^{\mathrm{K}^*}\right\} = \sum_{a=1}^{r-1}, \left\{d^{\mathrm{K}(a)}\right\}, \left\{d^{\mathrm{I}^*}\right\} = \sum_{a=2}^{r}\left\{d^{\mathrm{I}(a)}\right\}\right) \text{ with } \left\{d^{\mathrm{K}(a)}\right\} \in \text{Ker } A^{\mathrm{t}} \text{ and } \left\{d^{\mathrm{I}(a)}\right\} \in \text{Im } A$$

satisfying:

$$\left\{e^{(2)}\right\} = \{0\}, \left\{e^{(3)}\right\} = \{0\}, \ldots, \left\{e^{(r)}\right\} = \{0\} \tag{68}$$

The problem is to find if there exists displacements ($\{D^{\mathrm{K}}\}, \{D^{\mathrm{I}}\}$) (defined as follows), which cancel length variations at order $r+1$:

$$\exists\left(\left\{D^{\mathrm{K}}\right\}, \left\{D^{\mathrm{I}}\right\}\right) \in (\text{Ker } A^{\mathrm{t}} \times \text{Im } A), \quad \left\{e^{(r+1)}\right\} = \{0\} \tag{69}$$

with

$$\left\{d^{\mathrm{K}}\right\} = \sum_{a=1}^{r-1}\left\{d^{\mathrm{K}(a)}\right\} + \left\{D^{\mathrm{K}}\right\} = \left\{d^{\mathrm{K}^*}\right\} + \left\{D^{\mathrm{K}}\right\} \quad \text{where } \|D^{\mathrm{K}}\| \stackrel{0}{=} O_r \tag{70}$$

$$\left\{d^{\mathrm{I}}\right\} = \sum_{a=2}^{r}\left\{d^{\mathrm{I}(a)}\right\} + \left\{D^{\mathrm{I}}\right\} = \left\{d^{\mathrm{I}^*}\right\} + \left\{D^{\mathrm{I}}\right\} \quad \text{where } \|D^{\mathrm{I}}\| \stackrel{0}{=} O_{r+1} \tag{71}$$

where

$$\left\{D^{\mathrm{K}}\right\} = \sum_{\rho=1}^{m}\bar{\mu}_{\rho}\left\{d^{\mathrm{K}\rho}\right\} \quad \text{and} \quad \left\{D^{\mathrm{I}}\right\} = \sum_{p=1}^{r_A}U_p\left\{d^{\mathrm{I}p}\right\} \tag{72}$$

To establish, in accordance with the exact relationship (23), general formulations allowing the calculation of vector $\{e^{r+1}\}$ of length variation coefficients at order $r+1$, it is necessary to differentiate even and odd values of r.

5.4.1. For odd values of r ($r = 2\eta - 1$)

$$\begin{aligned}\left\{e^{r+1}\right\} = \left\{e^{(2\eta)}\right\} = {} & [A]^{\mathrm{t}}\left\{D^{\mathrm{I}}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{D^{\mathrm{K}}\right\} + \left[\Delta A_{d^{\mathrm{K}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(2\eta-2)}\right\} + \cdots \\ & + \left[\Delta A_{d^{\mathrm{K}(\eta-1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(\eta+1)}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{K}(\eta)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(\eta)}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2\eta-1)}\right\} + \cdots \\ & + \left[\Delta A_{d^{\mathrm{K}(2\eta-3)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(3)}\right\} + \left[\Delta A_{d^{\mathrm{K}(2\eta-2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2)}\right\} + \left[\Delta A_{d^{\mathrm{I}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2\eta-2)}\right\} + \cdots \\ & + \left[\Delta A_{d^{\mathrm{I}(\eta-1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(\eta+1)}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{I}(\eta)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(\eta)}\right\} \end{aligned} \quad (73)$$

The vector $\{e^{r+1}\}$ is a result of the summation of $2r$ terms as they are defined above.

5.4.2. For even values of r ($r = 2\eta$)

$$\begin{aligned}\left\{e^{(r+1)}\right\} = \left\{e^{(2\eta+1)}\right\} = {} & [A]^{\mathrm{t}}\left\{D^{\mathrm{I}}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{D^{\mathrm{K}}\right\} + \left[\Delta A_{d^{\mathrm{K}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(2\eta-1)}\right\} \\ & + \cdots + \left[\Delta A_{d^{\mathrm{K}(\eta)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(\eta+1)}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2\eta)}\right\} \\ & + \cdots + \left[\Delta A_{d^{\mathrm{K}(2\eta-1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2)}\right\} + \left[\Delta A_{d^{\mathrm{I}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2\eta-1)}\right\} + \cdots + \left[\Delta A_{d^{\mathrm{I}(\eta)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(\eta+1)}\right\} \end{aligned} \quad (74)$$

In this case, the vector $\{e^{(r+1)}\}$ is a result of the summation of $2r-1$ terms as they are defined above.

If the linear system $\{e^{(r+1)}\} = \{0\}$ (with b equations and N unknowns $\{\bar{\mu}\}$ and $\{U\}$ has no solution, the considered mechanism $\{d^{\mathrm{K}}\}$ (or $\{d^{\mathrm{K}^*}\}$) is of order r. On the other hand, if there exists at least one solution ($\{D^{\mathrm{K}}\}$, $\{D^{\mathrm{I}}\}$), associated to solutions ($\{\bar{\mu}\}$, $\{U\}$), corresponding mechanisms $\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}^*}\} + \{D^{\mathrm{K}}\}$ are of higher order than r. Its order determination needs to perform the $r+1$ step of the algorithm.

Remark. Nevertheless, if we work with arbitrary vectors of R^N (i.e. with $2N$ dependent variables), in each step the displacements which are solutions may be found. But systems to solve are then more important (more equations and more unknowns). Indeed, orthogonality between the two vectorial subspaces Ker A^{t} and Im A is used to take into account the link between variables. Then, the $m + r_A$ ($= N$) following relationships may be written:

$$\forall \rho = 1, \ldots, m: \quad \left\{d^{\mathrm{K}\rho}\right\} \bullet \left\{D^{\mathrm{I}}\right\} = 0 \quad (75)$$

$$\forall p = 1, \ldots, r_A: \quad \left\{d^{\mathrm{I}p}\right\} \bullet \left\{D^{\mathrm{K}}\right\} = 0 \quad (76)$$

where $\{d^{\mathrm{K}\rho}\}$ is the ρ^{nd} vector of the mechanism basis (Ker A^{t}), $\{d^{\mathrm{I}p}\}$ is the p^{nd} vector of the displacement basis, orthogonal to mechanisms (Im A) and "$\bullet$" is used for a scalar product.

These N relationships, associated with the b equations with $2N$ unknowns of system $\{e^{(r)}\} = \{0\}$ lead to solve a system (linear for $r \geq 2$) with $b + N$ equations and $2N$ unknowns.

6. Algorithm applications

This section is devoted to the utilisation of the algorithm in two situations: determination of mechanism's order and research of higher order mechanisms. Some examples illustrate this study.

6.1. Determination of mechanism's order on a simple example

We consider an assembly of four members constituting a T-shape (Fig. 3); order of mechanisms is required.

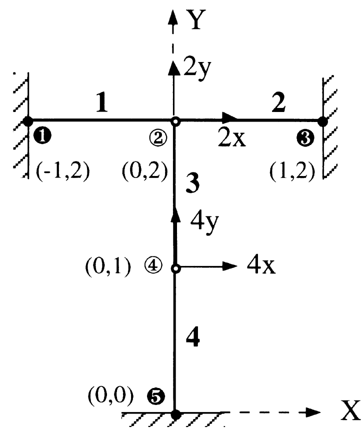

Fig. 3. Plane reticulated system with four members representing a T-shape, in its reference configuration.

This assembly has been studied by several authors, who do not agree together. For Connelly (1980) the infinitesimal mechanism is of order two, for Tarnai (1989) and Kuznetsov (1991a) it is of order three. For Salerno (1992), it is at least of the third order (because the corresponding program can not be used after order two).

Study of vectorial subspaces associated to equilibrium matrix $[A]$ of the assembly gives access to a mechanism basis $\{d^{K1}\} = \{0, 0, 1, 0\}^t$, and a basis of orthogonal displacements $\{d^{I1}\} = \{1, 0, 0, 0\}^t$, $\{d^{I2}\} = \{0, 1, 0, 0\}^t$ and $\{d^{I3}\} = \{0, 0, 0, 1\}^t$. The four components of these displacement vectors fit with the 4 dof ($2x$, $2y$, $4x$, $4y$) of this reticulated system.

Any mechanism $\{d^K\}$ and orthogonal displacement $\{d^I\}$ are given by:

$$\left\{d^K\right\} = \mu_1\left\{d^{K1}\right\} = \left\{0, 0, \mu_1, 0\right\}^t \tag{77}$$

and

$$\left\{d^I\right\} = u_1\left\{d^{I1}\right\} + u_2\left\{d^{I2}\right\} + u_3\left\{d^{I3}\right\} = \{u_1, u_2, 0, u_3\}^t \tag{78}$$

where μ_1, u_1, u_2 and u_3 are arbitrary real numbers.

We apply the algorithm to determine the order of the internal mechanism $\{d^K\}$ $(m = 1)$. Length variation coefficients vector $\{e^{(2)}\}$ is evaluated at order two.

$$\left\{e^{(2)}\right\} = [A]^t\left\{d^{I(2)}\right\} + \frac{1}{2}\left[\Delta A_{d^{K(1)}}\right]^t\left\{d^{K(1)}\right\} = \left\{u_1, \; -u_1, u_2 - u_3 + \frac{1}{2}\mu_1^2, u_3 + \frac{1}{2}\mu_1^2\right\}^t \tag{79}$$

In this example and following ones, notations may be simplified: order of scalars (μ_1, u_1, u_2 and u_3) is not indicated, since there is no possible confusion.

The system $\{e^{(2)}\} = \{0\}$ admits one solution:

$$u_1 = 0, u_2 = -\mu_1^2 \text{ and } u_3 = -\frac{1}{2}\mu_1^2 \tag{80}$$

Consequently, order of mechanism $\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\}$ is higher than one, since there exists a displacement $\{d^{\mathrm{I}(2)}\}$ (Fig. 4) which cancels at order two, the length variation coefficients for every member of the system.

Is this mechanism of order two ? Answer is given by the second step of algorithm.

Vector $\{e^{(3)}\}$ is calculated at order three, taking into account the following displacements:

$$\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\} \quad \left(\text{since } \{D^{\mathrm{K}}\} = \{0\} \text{ in case of } m = 1\right) \tag{81}$$

$$\{d^{\mathrm{I}}\} = \{d^{\mathrm{I}(2)}\} + \{D^{\mathrm{I}}\} \quad \text{with } \|D^{\mathrm{I}}\| \overset{0}{=} O_3 \tag{82}$$

where $\{d^{\mathrm{K}(1)}\} = \{0, 0, \mu_1, 0\}^{\mathrm{t}}$, $\{d^{\mathrm{I}(2)}\} = \{0, -\mu_1^2, 0, -(1/2)\mu_1^2\}^{\mathrm{t}}$ and $\{D^{\mathrm{I}}\} = \{U_1, U_2, 0, U_3\}^{\mathrm{t}}$.

Then:

$$\{e^{(3)}\} = [A]^{\mathrm{t}}\{D^{\mathrm{I}}\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} = \{U_1, -U_1, U_2 - U_3, U_3\}^{\mathrm{t}} \tag{83}$$

It is obvious that system $\{e^{(3)}\} = \{0\}$ admits only the zero solution:

$$U_1 = U_2 = U_3 = 0 \quad \text{or} \quad \{D^{\mathrm{I}}\} = \{d^{\mathrm{I}(3)}\} = \{0\} \tag{84}$$

There exists a displacement $\{D^{\mathrm{I}}\} = \{d^{\mathrm{I}(3)}\}$ which cancels $\{e\}$ at order three. Therefore, it may be concluded that $\{d^{\mathrm{K}}\}$ is a mechanism of order higher than two.

We evaluate $\{e^{(4)}\}$, with the following displacements:

$$\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\} \tag{85}$$

and

$$\{d^{\mathrm{I}}\} = \{d^{\mathrm{I}(2)}\} + \{d^{\mathrm{I}(3)}\} + \{D^{\mathrm{I}}\} \quad \text{with } \|D^{\mathrm{I}}\| \overset{0}{=} O_4 \tag{86}$$

Since (according to the expression (73))

$$\{e^{(4)}\} = [A]^{\mathrm{t}}\{D^{\mathrm{I}}\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{I}(2)}}\right]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} \tag{87}$$

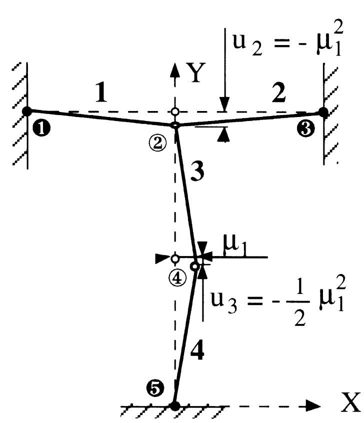

Fig. 4. Deformed reticulated system without length variations at orders one, two and three.

then

$$\left\{e^{(4)}\right\} = \left\{U_1 + \frac{1}{2}\mu_1^4,\ -U_1 + \frac{1}{2}\mu_1^4,\ U_2 - U_3 + \frac{1}{8}\mu_1^4,\ U_3 + \frac{1}{8}\mu_1^4\right\}^{\mathrm{t}} \tag{88}$$

It does not exist value of U_1 solution of the system $\{e^{(4)}\} = \{0\}$. Therefore, there is no displacement $\{d^{\mathrm{I}}\}$ cancelling at order four length variation generated by mechanism $\{d^{\mathrm{K}}\}$.

So, $\{d^{\mathrm{K}}\} = \{0,\ 0,\ \mu_1,\ 0\}^{\mathrm{t}}$ is an infinitesimal mechanism of order three, which is in accordance with results from Tarnai (1989) and Kuznetsov (1991a) and displacements expressions that were given by the first author.

It is interesting to notice that this method gives simultaneously the order of mechanism and values of node displacements ($\{d^{\mathrm{K}}\}$ and $\{d^{\mathrm{I}}\}$) which cancel length variations until the order of the mechanism (Fig. 4).

We notice that another T assembly, comprising five members (Fig. 5). which is considered by Kuznetsov (1988) as an order two mechanism, and by Salerno (1992) as a mechanism at *least* of order three, is also a mechanism of order three. Application of our algorithm leads to a system $\{e^{(4)}\} = \{0\}$, which has no U_1 solution.

6.2. Determination of mechanisms's order on two examples

6.2.1. Example with eight members given by Kuznetsov

Let us search mechanisms of order higher than one for a plane reticulated system submitted by Kuznetsov (1991a, 1991b, 1991c). Length of member 6 (Fig. 6) is parametered by "a" ($a \neq 0$).

It admits two internal independent mechanisms ($m = 2$):

$$\left\{d^{\mathrm{K1}}\right\} = \{0, 1, 0, 0, 0, 1, 0, 0\}^{\mathrm{t}} \quad \text{and} \quad \left\{d^{\mathrm{K2}}\right\} = \{0, 0, 0, 1, 0, 0, 0, 1\}^{\mathrm{t}} \tag{89}$$

The eight components of these vectors fit with the 8 dof ($2x$, $2y$, $3x$, $3y$, $6x$, $6y$, $7x$, $7y$) of the system. So, any mechanism $\{d^{\mathrm{K}}\}$ may be described by:

$$\left\{d^{\mathrm{K}}\right\} = \mu_1\left\{d^{\mathrm{K1}}\right\} + \mu_2\left\{d^{\mathrm{K2}}\right\} = \left\{0, \mu_1, 0, \mu_2, 0, \mu_1, 0, \mu_2\right\}^{\mathrm{t}} \tag{90}$$

with μ_1, μ_2 arbitrary real numbers.

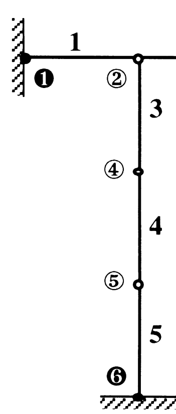

Fig. 5. Plane reticulated system with five members representing a T-shape, in its reference configuration.

An orthogonal displacement $\{d^{\mathrm{I}}\}$ is defined by:

$$\{d^{\mathrm{I}}\} = \{u_1, u_2, u_3, u_4, u_5, -u_2, u_6, -u_4\}^{\mathrm{t}} \tag{91}$$

with u_1, u_2, u_3, u_4, u_5 and u_6 being arbitrary real numbers.

The system admits higher order mechanism $\{d^{\mathrm{K}}\}$ if there exists no zero displacements $\{d^{\mathrm{I}(2)}\}$ such as $\{e^{(2)}\} = \{0\}$:

$$\{e^{(2)}\} = [A]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} + \frac{1}{2}[\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{d^{\mathrm{K}(1)}\} = \{0\} \Longrightarrow \left\{ \begin{array}{l} \mu_2 = a\mu_1 \\ (a-3)(a-1) = 0 \end{array} \right. \tag{92}$$

So, when length "a" of member 6 is not equal to 1 or 3, the system of equations has no solution, and consequently mechanisms are of order one.

For the two other cases ($a = 1$ or $a = 3$), system has a solution ($\{d^{\mathrm{K}(1)}\}$, $\{d^{\mathrm{I}(2)}\}$). All the mechanisms are of order one, except those of the main part of $\{d^{\mathrm{K}(1)}\}$ (with $\mu_2 = a\mu_1$) which are of order higher than one, i.e.:

$$\{d^{\mathrm{K}(1)}\} = \mu_1\{d^{\mathrm{K1}}\} + a\mu_1\{d^{\mathrm{K2}}\} = \{0, \mu_1, 0, a\mu_1, 0, \mu_1, 0, a\mu_1\}^{\mathrm{t}} \tag{93}$$

$$\{d^{\mathrm{I}(2)}\} = \left\{ -\frac{1}{4}\mu_1^2, 0, -\frac{1}{4}\mu_2^2, \frac{2-a}{8a}\mu_2^2, -\frac{1}{2}\mu_1^2, 0, -\frac{1}{2a}\mu_2^2, -\frac{2-a}{8a}\mu_2^2 \right\}^{\mathrm{t}} \quad \text{with } \mu_2 = a\mu_1 \tag{94}$$

Which is the order for these mechanisms (necessarily higher than one)?

6.2.1.1. Case 1: ($a = 1$). When length "a" of member 6 is equal to 1, according to relationship (92), all mechanisms are of first order, except those of the main part of $\{d^{\mathrm{K}(1)}\}$ defined by $\mu_2 = \mu_1 = \mu$:

$$\{d^{\mathrm{K}(1)}\} = \mu_1\{d^{\mathrm{K1}}\} + \mu_2\{d^{\mathrm{K2}}\} = \mu(\{d^{\mathrm{K1}}\} + \{d^{\mathrm{K2}}\}) = \{0, \mu, 0, \mu, 0, \mu, 0, \mu\}^{\mathrm{t}} \tag{95}$$

Are these mechanisms of order higher than two? Answer is given by applying the second step. So, we note:

$$\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\} + \{D^{\mathrm{K}}\} \quad \text{with } \|D^{\mathrm{K}}\| \stackrel{0}{=} O_2 \tag{96}$$

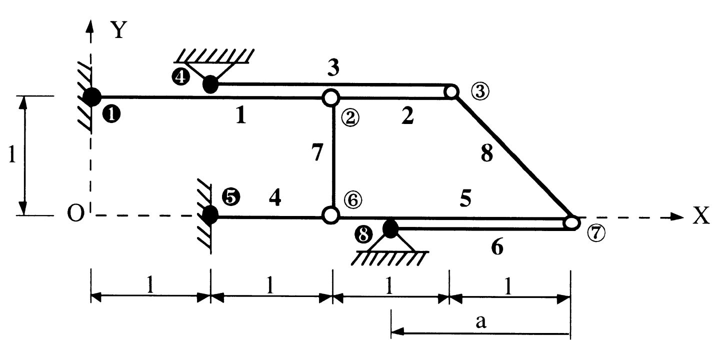

Fig. 6. Plane reticulated system with eight members studied by Kuznetsov, in its reference configuration.

and

$$\{d^{\mathrm{I}}\} = \{d^{\mathrm{I}(2)}\} + \{D^{\mathrm{I}}\} \quad \text{with } \|D^{\mathrm{I}}\| \overset{0}{=} O_3 \tag{97}$$

where

$$\{D^{\mathrm{K}}\} = \{0, \bar{\mu}_1, 0, \bar{\mu}_2, 0, \bar{\mu}_1, 0, \bar{\mu}_2\}^{\mathrm{t}}, \tag{98}$$

$$\{d^{\mathrm{I}(2)}\} = \left\{ -\frac{1}{4}\mu^2, 0, -\frac{1}{4}\mu^2, \frac{1}{8}\mu^2, -\frac{1}{2}\mu^2, 0, -\frac{1}{2}\mu^2, -\frac{1}{8}\mu^2 \right\}^{\mathrm{t}}, \tag{99}$$

$$\{D^{\mathrm{I}}\} = \{U_1, U_2, U_3, U_4, U_5, -U_2, U_6, -U_4\}^{\mathrm{t}}. \tag{100}$$

The system $\{e^{(3)}\} = [A]^{\mathrm{t}}\{D^{\mathrm{I}}\} + [\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{D^{\mathrm{K}}\} + [\Delta A_{d^{\mathrm{K}(1)}}]^{\mathrm{t}}\{d^{\mathrm{I}(2)}\} = \{0\}$ has no solution (refer to Vassart, 1995 for the related developments for this system and the subsequent).

Mechanisms of the main part $\{d^{\mathrm{K}(1)}\}$ $(= \mu(\{d^{\mathrm{K}1}\} + \{d^{\mathrm{K}2}\})$ are of order two and this system (with $a = 1$) has no mechanisms of order higher than two. This result is in agreement with assertions by Calladine and Pellegrino (1991b), and then Salerno (1992).

6.2.1.2. Case 2: ($a = 3$). When length "a" of member 6 is equal to 3, according to relationship (92), only the mechanisms of the main part of $\{d^{\mathrm{K}(1)}\}$ defined by $\mu_2 = 3\mu_1$ are of order higher than one:

$$\{d^{\mathrm{K}(1)}\} = \{0, \mu, 0, 3\mu, 0, \mu, 0, 3\mu\}^{\mathrm{t}} \tag{101}$$

We are searching the order for these mechanisms generated by $\{d^{\mathrm{K}(1)}\}$. We have to write:

$$\{d^{\mathrm{K}}\} = \{d^{\mathrm{K}(1)}\} + \{D^{\mathrm{K}}\} \quad \text{with } \|D^{\mathrm{K}}\| \overset{0}{=} O_2 \tag{102}$$

and

$$\{d^{\mathrm{I}}\} = \{d^{\mathrm{I}(2)}\} + \{D^{\mathrm{I}}\} \quad \text{with } \|D^{\mathrm{I}}\| \overset{0}{=} O_3 \tag{103}$$

where

$$\{d^{\mathrm{I}(2)}\} = \left\{ -\frac{1}{4}\mu^2, 0, -\frac{9}{4}\mu^2, -\frac{3}{8}\mu^2, -\frac{1}{2}\mu^2, 0, -\frac{3}{2}\mu^2, \frac{3}{8}\mu^2 \right\}^{\mathrm{t}}, \tag{104}$$

$$\{D^{\mathrm{I}}\} = \{U_1, U_2, U_3, U_4, U_5, -U_2, U_6, -U_4\}^{\mathrm{t}}, \tag{105}$$

$$\{D^{\mathrm{K}}\} = \{0, \bar{\mu}_1, 0, \bar{\mu}_2, 0, \bar{\mu}_1, 0, \bar{\mu}_2\}^{\mathrm{t}}. \tag{106}$$

We may assume that $\bar{\mu}_1 = 0$, and so simplify the developments since we are only interested in the relative displacement difference between the two independent mechanisms $\{d^{\mathrm{K}1}\}$ and $\{d^{\mathrm{K}2}\}$.

Solving $\{e^{(3)}\} = \{0\}$ leads to the following solution:

$$\bar{\mu}_2 = \frac{3}{8}\mu^2, \quad U_1 = 0, \quad U_2 = 0, \quad U_3 = 0, \quad U_4 = \frac{3}{8}\mu^3, \quad U_5 = 0, \quad U_6 = -\frac{3}{4}\mu^3 \tag{107}$$

Therefore, there exists displacements $\{D^{\mathrm{K}}\} = \{d^{\mathrm{K}(2)}\}$ and $\{D^{\mathrm{I}}\} = \{d^{\mathrm{I}(3)}\}$ which cancel length variation coefficients $\{e^{(3)}\}$ at order three:

$$\left\{D^{\mathrm{K}}\right\} = \left\{d^{\mathrm{K}(2)}\right\} = \left\{0, 0, 0, \frac{3}{8}\mu^2, 0, 0, 0, \frac{3}{8}\mu^2\right\}^{\mathrm{t}} \tag{108}$$

$$\left\{D^{\mathrm{I}}\right\} = \left\{d^{\mathrm{I}(3)}\right\} = \left\{0, 0, 0, \frac{3}{8}\mu^3, 0, 0, -\frac{3}{4}\mu^3, -\frac{3}{8}\mu^3\right\}^{\mathrm{t}} \tag{109}$$

Mechanism $\{d^{\mathrm{K}^*}\}$ is at least a mechanism of order three.

$$\left\{d^{\mathrm{K}^*}\right\} = \left\{d^{\mathrm{K}(1)}\right\} + \left\{d^{\mathrm{K}(2)}\right\} = \left\{0, \mu, 0, 3\mu + \frac{3}{8}\mu^2, 0, \mu, 0, 3\mu + \frac{3}{8}\mu^2\right\}^{\mathrm{t}} \tag{110}$$

Does the mechanism $\{d^{\mathrm{K}^*}\}$ lead again to mechanisms of higher order? In order to know it, we express:

$$\left\{d^{\mathrm{K}}\right\} = \left\{d^{\mathrm{K}(1)}\right\} + \left\{d^{\mathrm{K}(2)}\right\} + \left\{D^{\mathrm{K}}\right\} \quad \text{with } \|D^{\mathrm{K}}\| \overset{0}{=} O_3 \tag{111}$$

and

$$\left\{d^{\mathrm{I}}\right\} = \left\{d^{\mathrm{I}(2)}\right\} + \left\{d^{\mathrm{I}(3)}\right\} + \left\{D^{\mathrm{I}}\right\} \quad \text{with } \|D^{\mathrm{I}}\| \overset{0}{=} O_4 \tag{112}$$

where

$$\left\{D^{\mathrm{K}}\right\} = \left\{0, 0, 0, \bar{\mu}_2, 0, 0, 0, \bar{\mu}_2\right\}^{\mathrm{t}}, \tag{113}$$

$$\left\{D^{\mathrm{I}}\right\} = \{U_1, U_2, U_3, U_4, U_5, -U_2, U_6, -U_4\}^{\mathrm{t}}. \tag{114}$$

Length variation coefficients are evaluated at order four (according to expression (73)):

$$\begin{aligned}\left\{e^{(4)}\right\} = {} & [A]^{\mathrm{t}}\left\{D^{\mathrm{I}}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{D^{\mathrm{K}}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{K}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}(2)}\right\} + \left[\Delta A_{d^{\mathrm{K}(1)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(3)}\right\} \\ & + \left[\Delta A_{d^{\mathrm{K}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2)}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{I}(2)}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}(2)}\right\}\end{aligned} \tag{115}$$

It is possible to establish that system $\{e^{(4)}\} = \{0\}$ has no solution. Length variations cannot be cancelled at order four. Therefore, we may conclude that this system does not admit mechanism of order higher than $\{d^{\mathrm{K}^*}\}$ and this mechanism $\{d^{\mathrm{K}^*}\}$ is a mechanism of order three (Fig. 7), with:

$$\left\{d^{\mathrm{K}^*}\right\} = \left\{0, \mu, 0, 3\mu + \frac{3}{8}\mu^2, 0, \mu, 0, 3\mu + \frac{3}{8}\mu^2\right\}^{\mathrm{t}} \tag{116}$$

This is the same result that was proposed by Kuznetsov (1991b). Our algorithm gives equally the vector $\{d^{\mathrm{I}^*}\}$ associated to mechanism $\{d^{\mathrm{K}^*}\}$, i.e.:

$$\left\{d^{\mathrm{I}^*}\right\} = \left\{ -\frac{1}{4}\mu^2, 0, -\frac{9}{4}\mu^2, -\frac{3}{8}\mu^2 + \frac{3}{8}\mu^3, -\frac{1}{2}\mu^2, 0, -\frac{3}{2}\mu^2 - \frac{3}{4}\mu^3, \frac{3}{8}\mu^2 - \frac{3}{8}\mu^3 \right\}^{\mathrm{t}} \tag{117}$$

Therefore, this reticulated system, whose member 6 length is equal to three, admits mechanism vectors of order one, two and three. These different displacements can be described in the following manner:

- The mechanism $\{d^{\mathrm{K}}\}$ is a mechanism of order three if:

$$\left\{d^{\mathrm{K}}\right\} = \left\{d^{\mathrm{K}(1)}\right\} + \left\{d^{\mathrm{K}(2)}\right\} + \left\{D^{\mathrm{K}}\right\} \quad \text{with } \|D^{\mathrm{K}}\| \leq O_3 \tag{118}$$

where

$$\left\{d^{\mathrm{K}(1)}\right\} + \left\{d^{\mathrm{K}(2)}\right\} = \left\{0, \mu, 0, 3\mu + \frac{3}{8}\mu^2, 0, \mu, 0, 3\mu + \frac{3}{8}\mu^2\right\}^{\mathrm{t}}, \tag{119}$$

$$\left\{D^{\mathrm{K}}\right\} = \left\{0, \bar{\mu}_1, 0, \bar{\mu}_2, 0, \bar{\mu}_1, 0, \bar{\mu}_2\right\}^{\mathrm{t}}. \tag{120}$$

Or with an other writing:

$$\left\{d^{\mathrm{K}}\right\} \overset{(O_2)}{\approx} \left\{d^{\mathrm{K}(1)}\right\} + \left\{d^{\mathrm{K}(2)}\right\} \tag{121}$$

- The mechanism $\{d^{\mathrm{K}}\}$ is a mechanism of order two if:

$$\left\{d^{\mathrm{K}}\right\} = \left\{d^{\mathrm{K}(1)}\right\} + \left\{D^{\mathrm{K}}\right\} \quad \text{with } \|D^{\mathrm{K}}\| \leq O_2 \text{ and } \left\{D^{\mathrm{K}(2)}\right\} \neq \left\{d^{\mathrm{K}(2)}\right\} \tag{122}$$

where

$$\left\{d^{\mathrm{K}(1)}\right\} = \left\{0, \mu, 0, 3\mu, 0, \mu, 0, 3\mu\right\}^{\mathrm{t}}, \tag{123}$$

$$\left\{D^{\mathrm{K}}\right\} = \left\{0, \bar{\mu}_1, 0, \bar{\mu}_2, 0, \bar{\mu}_1, 0, \bar{\mu}_2\right\}^{t} \neq \left\{d^{\mathrm{K}(2)}\right\} = \left\{0, 0, 0, \frac{3}{8}\mu^2, 0, 0, 0, \frac{3}{8}\mu^2\right\}^{\mathrm{t}}. \tag{124}$$

- The mechanism $\{d^{\mathrm{K}}\}$ is a mechanism of order one if:

$$\left\{d^{\mathrm{K}}\right\} = \left\{D^{\mathrm{K}}\right\} \quad \text{with } \|D^{\mathrm{K}}\| = O_1 \text{ and } \left\{D^{\mathrm{K}(1)}\right\} \neq \left\{d^{\mathrm{K}(1)}\right\} \tag{125}$$

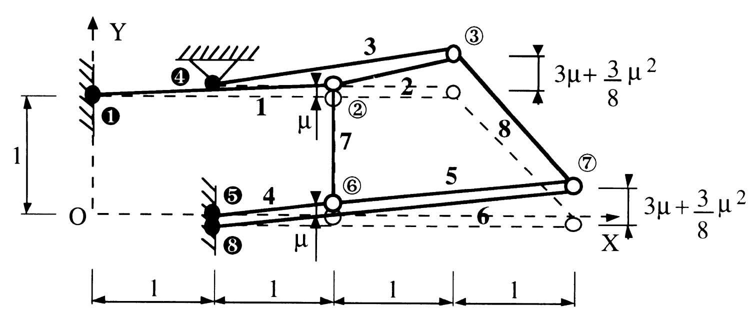

Fig. 7. Reticulated system with eight members deformed by its highest mechanism (order three) when the length of member 6 is equal to three.

where

$$\left\{D^{\mathrm{K}}\right\} = \left\{0, \bar{\mu}_1, 0, \bar{\mu}_2, 0, \bar{\mu}_1, 0, \bar{\mu}_2\right\}^t \neq \left\{d^{K(1)}\right\} = \left\{0, \mu, 0, 3\mu, 0, \mu, 0, 3\mu\right\}^{\mathrm{t}}. \tag{126}$$

Finally, this method gives access to the highest order of mechanisms for an assembly of members, and provides also the corresponding mechanism vector and those of intermediate orders. It allows to know all node displacements ($\{d^{\mathrm{K}^*}\}$ and $\{d^{\mathrm{I}^*}\}$)) associated with zero length variations until a given order.

6.2.2. Example with twelve members

We are now examining mechanisms of higher order, for a plane reticulated system with 12 members (Fig. 8) which combines the examples submitted by Tarnai (1989) and Kuznetsov (1991a, 1991b, 1991c). Length of member 6 is again parametered with "a" ($a \neq 0$).

This system admits three independent mechanisms (i.e. $m = 3$). Any mechanism $\{d^{\mathrm{K}}\}$ may be defined as follows with three scalars μ_1, μ_2 and μ_3:

$$\left\{d^{\mathrm{K}}\right\} = \left\{0, \mu_1, 0, \mu_2, 0, \mu_3, 0, \mu_1, 0, \mu_2, 0, 0\right\}^{\mathrm{t}} \tag{127}$$

The 12 components of this displacement vector are associated to the 12 dof ($2x$, $2y$, $3x$, $3y$, $4x$, $4y$, $6x$, $6y$, $7x$, $7y$, $10x$, $10y$) of this system.

Orthogonal displacements may be described by the following vector $\{d^{\mathrm{I}}\}$:

$$\left\{d^{\mathrm{I}}\right\} = \{u_1, u_2, u_3, u_4, u_5, 0, u_6, -u_2, u_7, -u_4, u_8, u_9\}^{\mathrm{t}} \tag{128}$$

where u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 and u_9 are arbitrary real values.

The first step of the algorithm allows to establish that the system $\{e^{(2)}\} = \{0\}$ admits non zero solutions (i.e. higher order mechanisms) only if parameter "a" is strictly positive and less than or equal to three (i.e. if $a \in]0, 3]$):

$$\left\{e^{(2)}\right\} = \{0\} \Longrightarrow \quad 3\mu_1^2 - 2\mu_1\mu_2 + \left(1 - \frac{2}{a}\right)\mu_2^2 = 0 \tag{129a}$$

$$\left\{e^{(2)}\right\} = \{0\} \Longrightarrow 3\mu_1^2 - 4\mu_1\mu_2 + \mu_2^2 + 2\mu_2\mu_3 - 3\mu_3^2 = 0 \tag{129b}$$

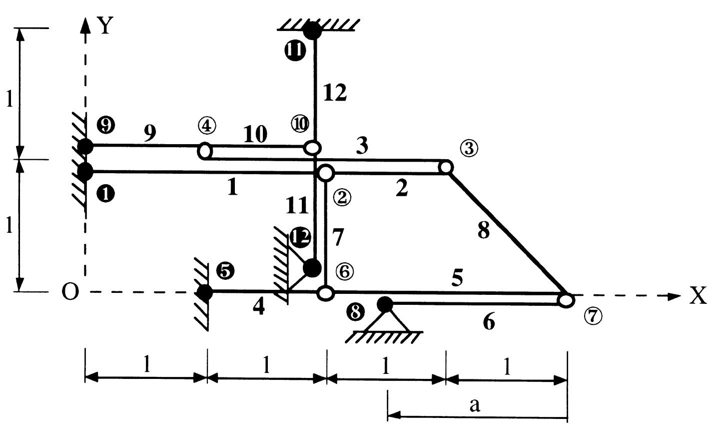

Fig. 8. Plane reticulated system with 12 members in its reference configuration.

We study, for some particular values of parameter "a", the order of mechanisms. For concision requirements, only mechanism vector solutions $\{d^{\mathrm{K}}\}$ of Ker A^{t}, are given, but associated vectors $\{d^{\mathrm{I}}\}$ (i.e. solution vectors $\{d^{\mathrm{I}}\}$ of Im A) are also a product of the algorithm.

6.2.2.1. Case 1: (a = 1). When length of member 6 is equal to one, the reticulated system admits four distinct mechanisms, whose order is higher than one:

- First mechanism:

$$\mu_1 = \mu, \mu_2 = \mu \quad \text{and} \quad \mu_3 = 0 \tag{130}$$

hence,

$$\left\{d^{\mathrm{K}(1)}\right\} = \left\{0, \mu, 0, \mu, 0, 0, 0, \mu, 0, \mu, 0, 0\right\}^{\mathrm{t}} \tag{131}$$

- Second mechanism:

$$\mu_1 = \mu, \mu_2 = \mu \quad \text{and} \quad \mu_3 = \frac{2}{3}\mu \tag{132}$$

hence,

$$\left\{d^{\mathrm{K}(1)}\right\} = \left\{0, \mu, 0, \mu, 0, \frac{2}{3}\mu, 0, \mu, 0, \mu, 0, 0\right\}^{\mathrm{t}} \tag{133}$$

- Third mechanism:

$$\mu_1 = \mu, \mu_2 = -3\mu \quad \text{and} \quad \mu_3 = 2\mu \tag{134}$$

hence,

$$\left\{d^{\mathrm{K}(1)}\right\} = \left\{0, \mu, 0, -3\mu, 0, 2\mu, 0, \mu, 0, -3\mu, 0, 0\right\}^{\mathrm{t}} \tag{135}$$

- Fourth mechanism:

$$\mu_1 = \mu, \mu_2 = -3\mu \quad \text{and} \quad \mu_3 = -4\mu \tag{136}$$

hence,

$$\left\{d^{\mathrm{K}(1)}\right\} = \left\{0, \mu, 0, -3\mu, 0, -4\mu, 0, \mu, 0, -3\mu, 0, 0\right\}^{\mathrm{t}} \tag{137}$$

For each of them, following steps of the algorithm have to be proceeded in order to calculate the order of mechanisms which are generated by this main part of $\{d^{\mathrm{K}(1)}\}$.

For the first mechanism vector $\{d^{\mathrm{K}(1)}\}$ (i.e. $\mu_1 = \mu, \mu_2 = \mu$ and $\mu_3 = 0$ or Eq. (131)), all equation systems admit until step 6, a solution (and only one). But at step 7, equation system $\{e^{(8)}\} = \{0\}$ has no solution with respect to U_9:

$$\left\{e^{(8)}\right\} \neq \{0\} \quad \text{because } e_{11}^{(8)} = U_9 + \frac{1}{512}\mu^8 \text{ and } e_{12}^{(8)} = -U_9 + \frac{1}{512}\mu^8 \tag{138}$$

Consequently, mechanism order for this system is seven. Corresponding mechanism vector $\{d^{\mathrm{K}}\}$ is given by mechanism vectors solution of preceding steps, that is:

$$\{d^{\mathrm{K}}\}\overset{(O_6)}{\approx}\{0,\mu,0,\mu+\bar{\mu}_2,0,2\bar{\mu}_2,0,\mu,0,\mu+\bar{\mu}_2,0,0\}^{\mathrm{t}} \tag{139}$$

with

$$\bar{\mu}_2=\frac{1}{8}\mu^2+\frac{3}{128}\mu^4+\frac{3}{256}\mu^5+\frac{27}{1024}\mu^6. \tag{140}$$

The three other mechanism vectors $\{d^{\mathrm{K}(1)}\}$, lead to three-order three mechanisms $\{d^{\mathrm{K}}\}$ ($\{e^{(4)}\}=\{0\}$ has no solution with respect to U_9):

$$\{d^{\mathrm{K}}\}\overset{(O_2)}{\approx}\left\{0,\mu,0,\mu+\frac{1}{8}\mu^2,0,\frac{2}{3}\mu-\frac{1}{12}\mu^2,0,\mu,0,\mu+\frac{1}{8}\mu^2,0,0\right\}^{\mathrm{t}} \tag{141}$$

$$\{d^{\mathrm{K}}\}\overset{(O_2)}{\approx}\left\{0,\mu,0,-3\mu-\frac{15}{8}\mu^2,0,2\mu+\frac{5}{4}\mu^2,0,\mu,0,-3\mu-\frac{15}{8}\mu^2,0,0\right\}^{\mathrm{t}} \tag{142}$$

$$\{d^{\mathrm{K}}\}\overset{(O_2)}{\approx}\left\{0,\mu,0,-3\mu-\frac{15}{8}\mu^2,0,-4\mu-\frac{15}{4}\mu^2,0,\mu,0,-3\mu-\frac{15}{8}\mu^2,0,0\right\}^{\mathrm{t}} \tag{143}$$

Finally, with $a=1$ (a being length of member 6) the mechanism of highest order found for this system is, therefore, a mechanism of order seven.

6.2.2.2. Case 2. ($a=3$). When length of member 6 is equal to three, the reticulated system admits two distinct mechanism vectors of order higher than one (since Eq. (129a) has a double root $\mu_2=3\mu_1$):

- First mechanism:

$$\mu_1=\mu,\ \mu_2=3\mu\quad\text{and}\quad\mu_3=2\mu \tag{144}$$

  hence,

$$\{d^{\mathrm{K}(1)}\}=\{0,\mu,0,3\mu,0,2\mu,0,\mu,0,3\mu,0,0\}^{\mathrm{t}} \tag{145}$$

- Second mechanism:

$$\mu_1=\mu,\ \mu_2=3\mu\quad\text{and}\quad\mu_3=0 \tag{146}$$

  hence,

$$\{d^{\mathrm{K}(1)}\}=\{0,\mu,0,3\mu,0,0,0,\mu,0,3\mu,0,0\}^{\mathrm{t}} \tag{147}$$

For these two mechanism vectors $\{d^{\mathrm{K}(1)}\}$, the different steps of algorithm are explicated, since the existence of a double root gives an infinity of solutions when the system admits at least one solution. As previously, we take $\bar{\mu}_1=0$, for each algorithm step, since only the relative difference between the three independent mechanism vectors has an interest. Consequently, the unknown vector $\{D^{\mathrm{K}}\}$ of Ker A^{t} is:

$$\{D^{\mathrm{K}}\}=\{0,0,0,\bar{\mu}_2,0,\bar{\mu}_3,0,0,0,\bar{\mu}_2,0,0\}^{\mathrm{t}} \tag{148}$$

Firstly, let us consider the first mechanism vector $\{d^{\mathrm{K}(1)}\}$ (i.e. $\mu_1=\mu$, $\mu_2=3\mu$ and $\mu_3=2\mu$ or Eq. (144)). Equation system $\{e^{(3)}\}=\{0\}$ of the second step admits an infinity of solutions, which can be parametered, for example, by $\bar{\mu}_3^{(2)}$:

$$\left\{e^{(3)}\right\} = \{0\} \Longrightarrow \bar{\mu}_2^{(2)} = \frac{3}{8}\mu^2 + \bar{\mu}_3^{(2)} \tag{149}$$

hence,

$$\left\{d^{\mathrm{K}(2)}\right\} = \left\{D^{\mathrm{K}(2)}\right\} = \left\{0, 0, 0, \frac{3}{8}\mu^2 + \bar{\mu}_3^{(2)}, 0, \bar{\mu}_3^{(2)}, 0, \mu, 0, \frac{3}{8}\mu^2 + \bar{\mu}_3^{(2)}, 0, 0\right\}^{\mathrm{t}} \tag{150}$$

On the other hand, equation system $\{e^{(4)}\} = \{0\}$ of the third step does not admit solution (in term of U_9) and this is true, whatever can be the value attributed to parameter $\bar{\mu}_3^{(2)}$:

$$\forall \bar{\mu}_3^{(2)}, \left\{e^{(4)}\right\} \neq \{0\} \quad \text{because } e_{11}^{(4)} = U_9 + 8\mu^4 \text{ and } e_{12}^{(4)} = -U_9 + 8\mu^4 \tag{151}$$

This mechanism vector $\{d^{\mathrm{K}(1)}\}$ leads to mechanisms $\{d^{\mathrm{K}}\}$ which are at most of order three and which can be defined as follows:

$$\left\{d^{\mathrm{K}}\right\} \overset{(O_2)}{\approx} \left\{0, \mu, 0, 3\mu + \frac{3}{8}\mu^2 + \bar{\mu}_3^{(2)}, 0, 2\mu + \bar{\mu}_3^{(2)}, 0, \mu, 0, 3\mu + \frac{3}{8}\mu^2 + \bar{\mu}_3^{(2)}, 0, 0\right\}^{\mathrm{t}} \quad \forall \bar{\mu}_3^{(2)} \tag{152}$$

Let us consider now, the second mechanism vector $\{d^{\mathrm{K}(1)}\}$ (i.e. $\mu_1 = \mu$, $\mu_2 = 3\mu$ and $\mu_3 = 0$ or Eq. (146)). Equation system $\{e^{(3)}\} = \{0\}$ of the second step, admits an infinity of solutions, which can be again parametered by $\bar{\mu}_3^{(2)}$:

$$\left\{e^{(3)}\right\} = \{0\} \Longrightarrow \bar{\mu}_2^{(2)} = \frac{3}{8}\mu^2 - 3\bar{\mu}_3^{(2)} \tag{153}$$

But the value of this parameter $\bar{\mu}_3^{(2)}$ is conditioned by the equation system $\{e^{(4)}\} = \{0\}$ of the third step of the algorithm. In order to have solutions for this last system, $\bar{\mu}_3^{(2)}$ has to be equal to $\mu^2/4$. In this case, this system has an infinity of solutions, which can be then parametered with $\bar{\mu}_3^{(3)}$:

$$\left\{e^{(4)}\right\} = \{0\} \Longrightarrow \bar{\mu}_3^{(2)} = \frac{1}{4}\mu^2 \text{ and } \bar{\mu}_2^{(3)} = -\frac{9}{4}\mu^3 - 3\bar{\mu}_3^{(3)} \tag{154}$$

Solving system $\{e^{(5)}\} = \{0\}$ of the fourth step does not require any restriction for this parameter $\bar{\mu}_3^{(3)}$. That is why, associated solutions may be parametered with two variables $\bar{\mu}_3^{(3)}$ and $\bar{\mu}_3^{(4)}$:

$$\left\{e^{(5)}\right\} = \{0\} \Longrightarrow \bar{\mu}_2^{(4)} = \frac{393}{128}\mu^4 - \frac{3}{2}\mu\bar{\mu}_3^{(3)} - 3\bar{\mu}_3^{(4)} \tag{155}$$

On the other hand, the equation system $\{e^{(6)}\} = \{0\}$ of the fifth step, admits solutions only if parameter $\bar{\mu}_3^{(3)}$ is equal to $-\mu^3$. The related infinity of solutions may be parametered with $\bar{\mu}_3^{(5)}$ (knowing that there are already parametered with $\bar{\mu}_3^{(4)}$:

$$\left\{e^{(6)}\right\} = \{0\} \Longrightarrow \bar{\mu}_3^{(3)} = -\mu^3 \text{ and } \bar{\mu}_2^{(5)} = -\frac{3225}{256}\mu^5 - \frac{3}{2}\mu\bar{\mu}_3^{(4)} - 3\bar{\mu}_3^{(5)} \tag{156}$$

Whatever can be values of the two parameters $\bar{\mu}_3^{(4)}$ and $\bar{\mu}_3^{(5)}$, the system $\{e^{(7)}\} = \{0\}$ of the sixth step admits an infinity of solutions, parametered with a new variable $\bar{\mu}_3^{(6)}$:

$$\left\{e^{(7)}\right\} = \{0\} \Longrightarrow \bar{\mu}_2^{(6)} = \frac{16707}{1024}\mu^6 + \frac{81}{8}\mu^2\bar{\mu}_3^{(4)} - \frac{3}{2}\mu\bar{\mu}_3^{(5)} - 3\bar{\mu}_3^{(6)} \tag{157}$$

Finally, for any value of the three parameters $\bar{\mu}_3^{(4)}$, $\bar{\mu}_3^{(5)}$ and $\bar{\mu}_3^{(6)}$, system $\{e^{(8)}\} = \{0\}$ of the seventh step admits no solution in function of U_9:

$$\forall \bar{\mu}_3^{(4)}, \bar{\mu}_3^{(5)}, \bar{\mu}_3^{(6)}, \left\{e^{(8)}\right\} \neq \{0\} \quad \text{because } e_{11}^{(8)} = U_9 + \frac{1}{512}\mu^8 \text{ and } e_{12}^{(8)} = -U_9 + \frac{1}{512}\mu^8 \tag{158}$$

For this reticulated system (in case of $a = 3$), highest mechanisms are of order seven. Related mechanism vectors may be described by the following vector $\{d^{\mathrm{K}}\}$, parametered with the three arbitrary variables $\bar{\mu}_3^{(4)}$, $\bar{\mu}_3^{(5)}$ and $\bar{\mu}_3^{(6)}$:

$$\left\{d^{\mathrm{K}}\right\} \overset{(O_6)}{\approx} \left\{0, \mu, 0, 3\mu + \bar{\mu}_2, 0, \bar{\mu}_3, 0, \mu, 0, 3\mu + \bar{\mu}_2, 0, 0\right\}^{\mathrm{t}} \tag{159}$$

with

$$\bar{\mu}_2 = -\frac{3}{8}\mu^2 + \frac{3}{4}\mu^3 + \frac{585}{128}\mu^4 - 3\bar{\mu}_3^{(4)} - \frac{3225}{256}\mu^5 - \frac{3}{2}\mu\bar{\mu}_3^{(4)} - 3\bar{\mu}_3^{(5)} + \frac{16707}{1024}\mu^6 + \frac{81}{8}\mu^2\bar{\mu}_3^{(4)} - \frac{3}{2}\mu\bar{\mu}_3^{(5)} - 3\bar{\mu}_3^{(6)}, \tag{160}$$

$$\bar{\mu}_3 = \frac{1}{4}\mu^2 - \mu^3 + \bar{\mu}_3^{(4)} + \bar{\mu}_3^{(5)} + \bar{\mu}_3^{(6)}. \tag{161}$$

Order seven is a relatively high order and, it could be thought to a certain extent that this reticulated system has a behaviour conditioned by a finite mechanism, even if it remains fundamentally different. Is it possible to differentiate very high-order mechanisms and finite mechanisms?

7. Finite mechanisms

7.1. Condition of existence of a finite mechanism

A reticulated system admits a finite mechanism, if there exists a non zero displacement which does not generate length variations of any order. A first determination method of finite mechanisms, for a kinematically and statically indeterminate system, could be to search if the exact relationship (23) admits a non zero solution, i.e. if:

$$\exists\left(\left\{d^{\mathrm{K}}\right\}, \left\{d^{\mathrm{I}}\right\}\right) \in (\mathrm{Ker}\, A^{\mathrm{t}} - \{0\} \times \mathrm{Im}\, A),$$

$$[A]^{\mathrm{t}}\left\{d^{\mathrm{I}}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{K}}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}}\right\} + \left[\Delta A_{d^{\mathrm{K}}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{I}}}\right]^{\mathrm{t}}\left\{d^{\mathrm{I}}\right\} = \{0\} \tag{162}$$

Equations of this system are quadratic, and except for very simple cases, it is difficult to solve it. Also, it is interesting to have a more simple method allowing to know if a reticulated system possesses finite mechanisms.

7.2. Utilisation of the algorithm to detect finite mechanisms

Our algorithm provides an alternative method for the finite mechanism determination, when it is associated with a stop criterion, which may be based on a conjecture, applicable for most of constructive reticulated system. In fact, a finite mechanism for an assembly, comprising a finite number of members, may be considered as a mechanism of infinite order. This explains why a stop criterion is needed, since the algorithm can not practically be applied until infinite order.

Stop criterion concerns connected reticulated systems with b members (b = finite number) without sliding node, with at least one fixed node. Length of members are finite, like node distances (i.e. of order O_0). This criterion is based on the following conjecture:

"For a reticulated system satisfying the above conditions and admitting only mechanisms, it may be asserted that:

$$\forall\|d\| = O_1 : \|e\| \geq O_r \text{ with } r = 2^{E(b/2)} \tag{163}$$

where d is an arbitrary displacement of order one, b is a finite number of members, E(b/2) is the integer part of b/2."

According to our conjecture, if at step $2^{E(b/2)} - 1$ of the algorithm, there always exists displacements which cancel length variations of order $2^{E(b/2)}$, then the reticulated system admits a finite mechanism (its development is known until this order).

This conjecture is based on following non exhaustive arguments:

Firstly, infinitesimal mechanisms for reticulated systems with two or three members ($b = 2$ or $b = 3$), satisfying the above conditions, are associated with length variations of order two. They are associated with length variations of order four or less, for a system with four members (the given example in Section 6.1 admits this maximal order, i.e. $\forall\|d\| = O_1$, $\min\|e\| = O_4$).

Secondly, if infinitesimal mechanisms of a b members system are associated with length variations of maximum order $2^{E(b/2)}$, then the way to have a system, without finite mechanisms, with length variations of higher maximum order, would be to add two supplementary members. In fact, length variation order may be double with two supplementary aligned members. We have so a system, with $b + 2$ members, admitting infinitesimal mechanisms associated with length variations of maximum order equal to $2^{E(b/2)+1}$.

Tarnai's example (Tarnai, 1989) (Fig. 9) corresponds to the limit case, of a b members system which admits an infinitesimal mechanism associated with length variations exactly equal to $2^{E(b/2)}$. This corresponds, according to the conjecture, to the infinitesimal mechanism of highest order admissible by a reticulated connected system with b members, without sliding nodes and with finite length of members and distance between nodes (i.e. order O_0).

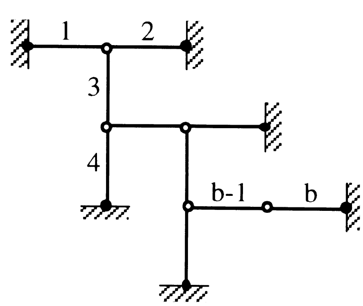

Fig. 9. Plane reticulated system with b members admitting infinitesimal mechanism of "$2^{E(b/2)}$" order.

7.3. Application of the algorithm stop criterion for a simple example

Let us study mechanisms for a plane reticulated system comprising three aligned members (Fig. 10), whose relative position of nodes 1, 2, 3 and 4 is respectively parametered by a_1, a_2 and a_3 (with $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$).

Study of equilibrium matrix $[A]$ of this reticulated system in its reference state, gives access to a mechanism basis $\{d^{\mathrm{K}}\}$ and an orthogonal displacement basis $\{d^{\mathrm{I}}\}$:

$$[A] = \begin{bmatrix} a_1 & -a_2 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & -a_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 2x \\ 2y \\ 3x \\ 3y \end{Bmatrix} \Longrightarrow \{d^{\mathrm{K1}}\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \{d^{\mathrm{K2}}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \text{ and } \{d^{\mathrm{I1}}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \{d^{\mathrm{I2}}\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tag{164}$$

These mechanisms are described with the following vector $\{d^{\mathrm{K}}\}$:

$$\{d^{\mathrm{K}}\} = \mu_1\{d^{\mathrm{K1}}\} + \mu_2\{d^{\mathrm{K2}}\} = \{0, \mu_1, 0, \mu_2\}^{\mathrm{t}} \tag{165}$$

While, orthogonal displacements can be described by vector $\{d^{\mathrm{I}}\}$:

$$\{d^{\mathrm{I}}\} = u_1\{d^{\mathrm{I1}}\} + u_2\{d^{\mathrm{I2}}\} = \{u_1, 0, u_2, 0\}^{\mathrm{t}} \tag{166}$$

According to our conjecture, this three member system admits a finite mechanism, if at step 1 (i.e. $2^{\mathrm{E}(b/2)} - 1 = 2^1 - 1 = 1$) of the algorithm, it exists non zero displacements which cancel order two length variations, i.e.:

$$\{d^{\mathrm{K}}\} = \text{finite mechanism if } \exists\left(\{d^{\mathrm{K}(1)}\}, \{d^{\mathrm{I}(2)}\}\right) \in (\mathrm{Ker}\, A^{\mathrm{t}} - \{0\} \times \mathrm{Im}\, A) : \{e^{(2)}\} = \{0\} \tag{167}$$

So, the first step of the algorithm has solely to be applied to determine if the system admits finite mechanisms or only infinitesimal mechanisms of order one, when application conditions for the stop criterion are satisfied (particularly finite distances between nodes).

Then, vector $\{e^{(2)}\}$ (length variation of order two) is calculated:

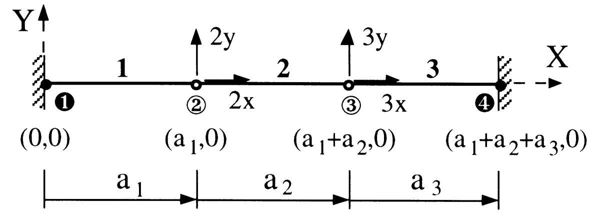

Fig. 10. Plane reticulated system with three aligned members.

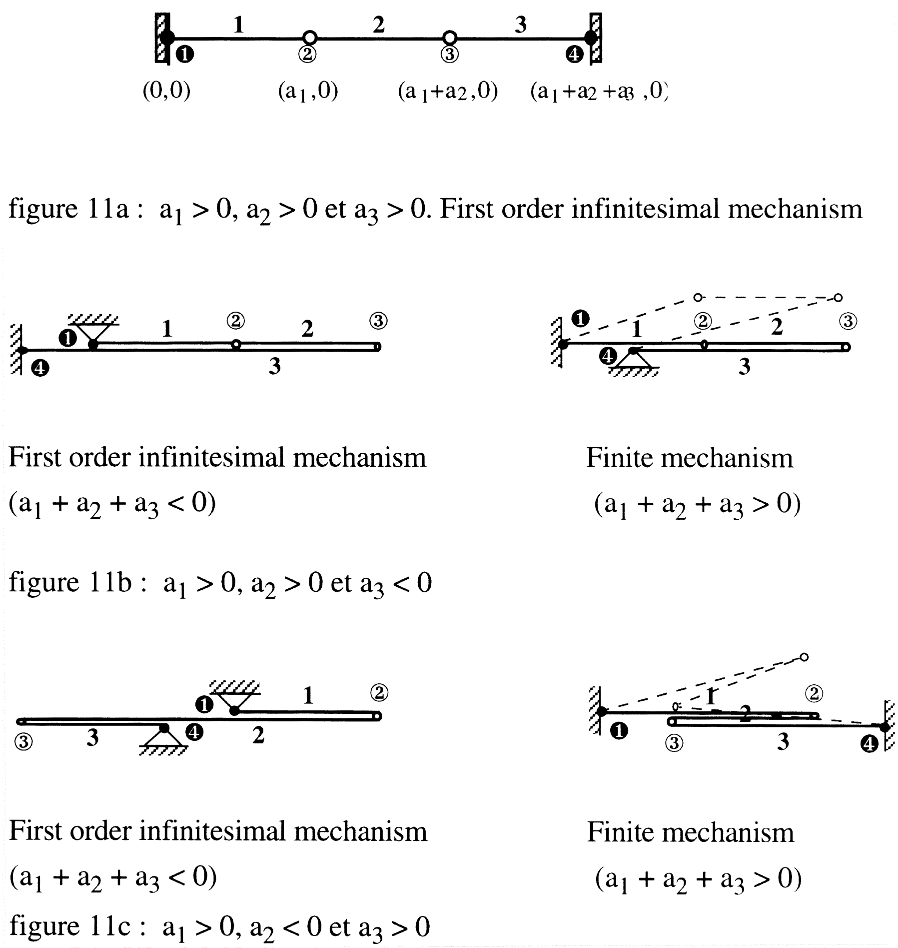

Fig. 11. Type of mechanisms of the system with three aligned members according to the relative position of its nodes.

$$\left\{e^{(2)}\right\} = [A]^{\mathrm{t}}\left\{d^{1}\right\} + \frac{1}{2}\left[\Delta A_{d^{\mathrm{K}}}\right]^{\mathrm{t}}\left\{d^{\mathrm{K}}\right\} = \left\{a_1 u_1 + \frac{1}{2}\mu_1^2,\; -a_2 u_1 + a_2 u_2 + \frac{1}{2}(\mu_1 - \mu_2)^2,\; -a_3 u_2 + \frac{1}{2}\mu_2^2\right\}^{\mathrm{t}} \tag{168}$$

System $\{e^{(2)}\} = \{0\}$ admits a non zero solution, only if the equation admits also a non zero solution

$$\left\{e^{(2)}\right\} = \{0\} \Longrightarrow \frac{\mu_1^2}{a_1} + \frac{(\mu_1 - \mu_2)^2}{a_2} + \frac{\mu_2^2}{a_3} = 0 \tag{169}$$

It can be already deduced that there is no non zero solution in case of same sign parameters a_1, a_2 and a_3. Consequently, if parameters a_1, a_2 and a_3 are, all together, either positive or negative (which leads to the same geometry) then the reticulated system (Fig. 11(a) possesses only infinitesimal mechanisms of order one:

$$\text{If } a_1 > 0, a_2 > 0, a_3 > 0 \text{ or if } a_1 < 0, a_2 < 0, a_3 < 0 \Longrightarrow \left\{e^{(2)}\right\} \neq \{0\} \quad \forall(\mu_1, \mu_2) \neq (0, 0) \tag{170}$$

According to symmetry statements, other cases can be related to the study of the two following cases:

$$a_1 > 0, a_2 > 0, a_3 < 0 \quad \text{and} \quad a_1 > 0, a_2 < 0, a_3 > 0 \tag{171}$$

Preceding Eq. (169) has a non zero solution, only if its reduced discriminant Δ' is positive or null:

$$\frac{\mu_1^2}{a_1} + \frac{(\mu_1 - \mu_2)^2}{a_2} + \frac{\mu_2^2}{a_3} = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\mu_1^2 - \frac{2\mu_1\mu_2}{a_2} + \left(\frac{1}{a_2} + \frac{1}{a_3}\right)\mu_2^2 = 0 \Longrightarrow \Delta' = -\frac{a_1 + a_2 + a_3}{a_1 a_2 a_3} \tag{172}$$

In fact, for the two studied cases, the product $a_1 a_2 a_3$ is strictly negative, and the reduced discriminant Δ' is of the same sign as the sum of the three parameters:

$$\Delta' \geq 0 \;\Leftrightarrow\; a_1 + a_2 + a_3 \geq 0 \tag{173}$$

When this condition is satisfied (i.e. when $a_1 + a_2 + a_3 \geq 0$), there exists non zero displacements such as length variations are equal to zero until order two, and consequently, the system admits mechanisms with order higher than one. That is why, when application conditions are satisfied, it is not necessary to proceed further calculations, since, according to the stop criterion, the system admits a finite mechanism (Fig. 11(b) and (c)). On the other hand, if the relationship is not verified, then the system admits only order one infinitesimal mechanisms (Fig. 11(b) and (c)).

8. Comments

Thanks to geometrical or energetic characterisation it may be simply determined if infinitesimal mechanisms are of first order or order higher than one. What we call energetic characterisation test is equivalent to the one given by Calladine and Pellegrino (1991a, 1991b, 1992)). However, geometrical test which is the first step of the submitted algorithm, provides also the displacements of the nodes associated with mechanisms of order higher than one.

Moreover, one has now an analytic matrix method of determination of order for any mechanism of a reticulated system (even when some lengths are parametered). When compared with the method given by Tarnai (1989), this method has the advantage to give a direct access to solution displacements. It has also the advantage, when compared with the method submitted by Salerno (1992), to give solution displacements in analytic form and not in numerical form. Moreover, some other advantages of our method may be underlined. We work directly on member length variations $\{e\}$, not on the deformation energy for which $\{e\}$ appears under a quadratic form, which requires to double the number of necessary developments. We also use explicitly basis of vectorial subspaces Ker A^{t} and Im A, and consequently, for each step we deal with a fixed number of vectors; with Salerno's method the number of vectors is greater at each step (m, $m(m+1)/2$, ...).

The relative simplicity of our method (that can always lead to the resolution of linear equation systems) allows to process, without the assistance of a computer, most of the kinematically indeterminate and relatively complex systems. For truly complex assemblies (admitting many degrees of freedom and/or members), a small program can be realised.

In summary, the submitted algorithm possesses a double interest: it determines the order of a given mechanism and also gives all the distinct mechanism vectors for a reticulated system. Lastly, with the submitted stop criterion, this algorithm identifies finite mechanisms.

References

Calladine, C.R., Pellegrino, S., 1991a. First-order infinitesimal mechanisms. Int. J. Solids Structures 27, 505–515.

Calladine, C.R., Pellegrino, S., 1991b. Authors' closure. Int. J. Solids Structures 27, 521–522.

Calladine, C.R., Pellegrino, S., 1992. Further remarks on first-order infinitesimal mechanisms. Int. J. Solids Structures 29, 2119–2122.

Connelly, R., 1980. The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces. Advances in Mathematics 37, 272–299.

Koiter, W.T., 1984. On Tarnai's conjecture with reference to both statically and kinematically indeterminate structures. Report No. 788 of Laboratory for Engineering Mechanics, Delft.

Kuznetsov, E.N., 1988. Underconstrained structural systems. Int. J. Solids Structures 24, 153–163.

Kuznetsov, E.N., 1991a. Systems with infinitesimal mobility. Part I: Matrix analysis and first-order mechanisms. J. Appl. Mech 58, 513–519.

Kuznetsov, E.N., 1991b. Systems with infinitesimal mobility. Part II: Compound and higher order mechanisms. J. Appl. Mech 58, 520–526.

Kuznetsov, E.N., 1991c. Underconstrained Structural Systems. Springer, New York.

Pellegrino, S., Calladine, C.R., 1986. Matrix analysis of statically and kinematically indeterminate frameworks. Int. J. Solids Structures 22, 409–428.

Salerno, G., 1992. How to recognise the order of infinitesimal mechanisms: a numerical approach. Int. J. Number. Methods Eng 35, 1351–1395.

Sheck, H.J., 1974. The force density method for formfinding and computation of general networks. Computer Methods in Applied Mechanics and Engineering 3, 115–134.

Tarnai, T., 1984. Comments on Koiter's Classification of Infinitesimal Mechanisms. Hung. Inst. Build. Sci, Budapest.

Tarnai, T., 1989. Higher order infinitesimal mechanisms. Acta Tech. Acad. Sci. Hung 102, 363–378.

Vassart, N., 1995. Ordre des mécanismes des systèmes réticulés cinématiquement indéterminés. Note Interne LMGC-95/CS-1. Université Montpellier II.